CAUSALITY CONDITIONS IN QUANTUM FIELD THEORY

V. Ya. FAĬNBERG

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor January 4, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 1758-1767 (June, 1961)

It is shown that the Bogolyubov causality condition and the condition of local commutativity of the operators $\varphi(x)$ of an interacting field lead to identical expressions for the matrix elements of the S matrix on the mass shell. In the study of this problem the necessary and sufficient conditions are found for the local solubility of the equation for the Heisenberg operators $\varphi(x)$ on the assumption that the commutator of the currents, $[j(x), j(y)]$, vanishes outside the light cone; that is, the so-called Wightman problem is solved.

1. INTRODUCTION

THE fundamental problem of relativistic quantum field theory is that of finding the properties of the matrix elements of the S matrix on the mass shell of real particles, i.e., for free-particle values of the four-momenta of all incident and scattered particles, $p_i^2 = m_i^2$ (here $p_i^2 = (p_i^0)^2 - (\mathbf{p}_i)^2$, and i indicates the type of particle). In the case of the Lagrangian formulation these properties are a consequence of the Heisenberg equations for the operators of the interacting field and the commutation relations. For a formulation of the local properties of the theory without the assumption of the existence of a Lagrangian (i.e., a dynamical principle), however, it is necessary to go outside the limits of the mass shell in the S matrix. In the axiomatic approach there are two main methods for accomplishing such an extrapolation.

The first method was proposed by Bogolyubov[1,2] and is as follows.* The S matrix is represented in the form of a functional of the free-field operators $[\varphi_{in}(x)$ and $\varphi_{out}(x)]$

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n S_n(x_1, \ldots x_n) : \varphi_{in}(x_1) \ldots \varphi_{in}(x_n): . \quad (1)$$

Here the colon indicates the normal product. For simplicity we shall consider the case of a neutral scalar field of mass m and assume that it does not form any bound states. The operators $\varphi_{in,out}(x)$ satisfy the homogeneous Klein-Gordon equation

$$(\Box - m^2)\, \varphi_{in,\, out}(x) = 0 \quad (2)$$

and obey the commutation relations

$$[\varphi_{in,\, out}(x),\ \varphi_{in,\, out}(y)] = -i\Delta(x - y,\ m)\,. \quad (3)$$

The S matrix is assumed unitary,

$$S^+S = 1, \quad (4)$$

and we have

$$\varphi_{out}(x) = S^+\varphi_{in}(x)\, S. \quad (5)$$

For the passage beyond the mass shell one defines the variational derivatives

$$\delta S/\delta\varphi_{in}(x) \equiv (\delta S(\eta)/\delta\eta(x))_{\eta=0}.$$

Here $S(\eta)$ is obtained from S by the replacement $\varphi_{in}(x) \to \varphi_{in}(x) + \eta(x)$, where $\eta(x)$ is an arbitrary external field.

Even though this departure from the mass shell is accomplished with preservation of Lorentz invariance, unitarity, and other necessary symmetry properties, it does not determine $S_n(x_1, \ldots x_n)$ unambiguously. To secure uniqueness and bring out the local properties of $S_n(x_1, \ldots x_n)$ one formulates the causality condition in the Bogolyubov method in the form

$$\delta j(x)/\delta\varphi_{in}(y) = 0, \qquad x \lesssim y \quad (6)$$

$x \lesssim y$ means that $x_0 < y_0$ or $(x-y)^2 < 0$, and $x \sim y$ means that $(x-y)^2 < 0$.) The current $j(x)$ is defined in the following way:

$$j(x) = iS^+ \frac{\delta S}{\delta\varphi_{in}(x)} = i\frac{\delta S}{\delta\varphi_{out}(x)} S^+. \quad (7)$$

In the other method, that of LSZ,[3,4] the current operator is also defined by means of Eq. (7).* The

*A detailed formulation of the initial axioms in the two methods can be found in papers by Bogolyubov and his coworkers[2] and by Lehmann, Symanzik, and Zimmermann (hereafter for brevity called LSZ).[3,4] Here we are interested mainly in the difference between the two methods in the formulation of the local properties of the theory.

*We disregard differences between the two approaches that are not important in this problem.[4]

interacting-field operator $\varphi(x)$ is introduced in terms of $j(x)$ as the retarded solution of the inhomogeneous Klein-Gordon equation

$$\varphi(x) = \varphi_{in}(x) - \int_{-\infty}^{+\infty} \Delta^{R}(x - x', m)\, j(x')\, d^4x' \qquad (8)$$

(the advanced solution is written in an analogous way). The requirement of causality, instead of being written as in Eq. (6), is formulated in this method as the vanishing of all the commutators of $\varphi(x)$ and $j(x)$ outside the light cone; that is, for $x \sim y$ we must have

$$[j(x), j(y)] = 0, \qquad (9a)$$

$$[\varphi(x), j(y)] = 0, \qquad (9b)$$

$$[\varphi(x), \varphi(y)] = 0. \qquad (9c)$$

In addition, in both methods one assumes the stability of the vacuum state and of the one-particle states.

It is not obvious that these two methods of going beyond the mass shell for the determination of the local properties of the S matrix are identical. Moreover, it has been asserted (cf. e.g., references 5, 6) that there are differences in principle between the two approaches (cf. end of Sec. 3).

The main purpose of the present paper is to show, without bringing in supplementary hypotheses (of the type of the adiabatic hypothesis) that the two methods are equivalent from the point of view of the properties of the matrix elements of the S matrix on the mass shell, but lead in general to different $S_n(x_1, \ldots x_n)$. It is important to emphasize that in the solution of this problem the need does not arise to make special use of the asymptotic conditions of LSZ.[3] The paper also contains the necessary and sufficient conditions for the existence of a causal [in the sense of Eqs. (9a) – (9c)] solution of Eq. (8) on the assumption that $j(x)$ obeys the condition (9a), without resort to the S matrix and to the connection (7) of the current $j(x)$ with S. This is the solution of a problem posed by Wightman.[7]

2. THE CONDITIONS FOR THE EXISTENCE OF LOCAL SOLUTIONS OF THE EQUATION (8)

In this section we shall study the local properties of Eq. (8), assuming that all of the operators that appear are the renormalized ones. Before proceeding to an exact statement of the problem, let us make two important remarks. Equation (8) has a number of remarkable properties.

First, owing to the relation

$$(\Box - m^2)\varphi(x) = j(x), \qquad (10)$$

it follows from (8) that if $[\varphi(x), \varphi(y)] = 0$ for $x \sim y$, then $[j(x), j(y)]$ also vanishes outside the cone, $(x-y)^2 < 0$; that is, the local property of $j(x)$ follows from that of $\varphi(x)$. The converse is in general untrue: the vanishing outside the light cone of the bracket $[j(x), j(y)]$ is not a sufficient condition for the local character of $\varphi(x)$. This fact will be of important use in what follows.

Second, the adiabatic hypothesis, which has been brought in particularly by Kaschluhn,[6] is in contradiction with Eq. (8) (on this point see papers by Haag, Hall and Wightman, and also Greenberg[8]). Moreover, even if this were not so, the adiabatic condition [in the sense of strong convergence: $\lim [\varphi(x) - \varphi_{in}(x)] = 0$ for $x_0 \to -\infty$] would be an additional assumption on the existence of a unitary connection between $\varphi(x)$ and $\varphi_{in}(x)$, which does not contain the assumption of the local character of $\varphi(x)$ [in the sense of Eq. (9)] and contains an implicit assumption of the finiteness of the renormalization in the theory.* Therefore we shall not resort to the adiabatic hypothesis.

Let us now state in its most general form the problem of the local properties of (8). In this connection we shall not use the connection (7) of the current $j(x)$ with the S matrix, and shall determine the necessary and sufficient conditions to be satisfied by the operator $j(x)$ in order for the solution of (8) for $\varphi(x)$ to be local, on the assumption that $j(x)$ commutes outside the light cone. This statement of the problem is equivalent to the Wightman problem.[7]

First we shall show (for later use) that the Bogolyubov causality condition (6), together with Eq. (7), is sufficient for the local properties (9a) – (9c) of the operators $\varphi(x)$ and $j(x)$ to hold. In fact, it follows from Eqs. (6) and (7) that

$$\delta j(x)/\delta\varphi_{in}(y) - \delta j(y)/\delta\varphi_{in}(x) = -i\,[j(x), j(y)] = 0, \quad x \sim y, \qquad (11)$$

$$\delta j(x)/\delta\varphi_{in}(y) = -i\theta(x_0 - y_0)\,[j(x), j(y)], \qquad (12)$$

apart from quasi-local terms, which will always be omitted in what follows.

*Finally, there is an argument against the introduction of the adiabatic hypothesis in the original idea of formulating the theory in the language of matrix elements of the S matrix, which from the very beginning avoids as far as possible all "unobservable" quantities such as unrenormalized operators, masses, charges, and so on.

Let us now take the simplest commutator $[\varphi(x), j(y)]$. Using Eqs. (8), (11), and (12) and the formula*

$$[\varphi_{in}(x), j(y)] = -i\int \Delta(x-x', m)\frac{\delta j(y)}{\delta\varphi_{in}(x')}d^4x', \quad (13)$$

we transform $[\varphi(x), j(x)]$ to the form†

$$[\varphi(x), j(y)] = -i\int \Delta(x-x', m)\frac{\delta j(y)}{\delta\varphi_{in}(x')}d^4x' - \int \Delta^R(x-x', m)[j(x'), j(y)]d^4x'$$
$$= -\int \Delta^R(x-x', m)\,\theta(x_0'-y_0)[j(x'), j(y)]d^4x' - \int \Delta^A(x-x', m)\,\theta(y_0-x_0')[j(x'), j(y)]d^4x. \quad (14)$$

From this expression it can be seen that $[\varphi(x), j(x)] = 0$ when $x \sim y$, since the first and second terms on the right in Eq. (14) contribute only inside the upper and lower light cones, respectively. In an entirely analogous way it can be shown that $[\varphi(x), \varphi(y)] = 0$ when $x \sim y$.

Let us now return to the problem stated earlier. The causality condition (6) suggests that in the general case it is convenient to try to find $\delta j(x)/\delta\varphi_{in}(y)$ in the form

$$\delta j(x)/\delta\varphi_{in}(y) = -i\theta(x_0-y_0)[j(x), j(y)] + \Lambda(x, y), \quad (15)$$

where $\Lambda(x,y)$ is an as yet arbitrary operator, whose properties must be established on the basis of the requirement (9b) of commutativity of the operators $\varphi(x)$ and $j(x)$ outside the light cone. Using (13) and (15), we get

$$[\varphi(x), j(y)] = -i\int \Delta(x-x', m)\Lambda(y, x')d^4x' - \int \Delta^R(x-x', m)\theta(x_0'-y_0)[j(x'), j(y)]d^4x' - \int \Delta^A(x-x', m)\theta(y_0-x_0')[j(x'), j(y)]d^4x'. \quad (16)$$

Unlike (14), Eq. (16) has on the right an additional operator term

$$F(x, y) \equiv -i\int \Delta(x-x', m)\Lambda(y, x')d^4x'. \quad (17)$$

In the spacelike region $x \sim y$ the second and third terms in (16) drop out for the same reasons as in the case of (14). From this it immediately follows that the vanishing of $F(x,y)$ outside the cone $(x-y)^2 = 0$ is the necessary and sufficient condition for the existence of a causal solution of (8) [causal in the sense of (9a) and (9b)].‡

*This formula is a consequence of Eq. (3) and the assumption that all operators are functionals of $\varphi_{in}(x)$ [or of $\varphi_{out}(x)$].

†We have used the relation $\Delta(x, m) = \Delta^R(x, m) - \Delta^A(x, m)$.

‡Naturally we assume that the solution in the form (8) exists, i.e., that the integral in (8) converges for an arbitrary matrix element of $j(x)$.

It is convenient to consider instead of $\Lambda(x,y)$ an arbitrary matrix element $\langle p|\Lambda(x,y)|p'\rangle$ between states with total four-momenta p and p′. These states are also characterized by the momenta of the particles that occur in them. Invariance under translations gives

$$\langle p|\Lambda(x, y)|p'\rangle = \exp(iQy)\langle p|\Lambda(x-y, 0)|p'\rangle \equiv \exp(iQy)\,\lambda(x-y), \quad (18)$$

where $Q = p - p'$. (We do not write out the other variables on which $\lambda(x)$ depends.) Introducing the notation

$$\langle p|F(x, y)|p'\rangle \equiv \exp(iQy)\, f(x-y),$$

we get instead of (7)

$$f(x) = \int \Delta(x-x', m)\,\lambda(x')\,d^4x'. \quad (19)$$

We shall now show that $f(x)$ vanishes outside the light cone $(x^2 < 0)$ if and only if the Fourier transform $\tilde{\lambda}(k)$ of the function $\lambda(x)$ is a polynomial of finite degree in k on both sheets of the hyperboloid

$$k^2 - m^2 = 0. \quad (20)$$

The sufficiency of the condition is easily proved. By hypothesis, on the hyperboloid (20)

$$\tilde{\lambda}(k) = \sum_{l=0}^{n} \tilde{\lambda}_l P_l(k), \quad (21)$$

where $P_l(k)$ are polynomials in k,* and the coefficients $\tilde{\lambda}_l$ do not depend on k.

Substituting (21) in (19), we at once find that

$$f(x) = \sum_{l=0}^{n} \tilde{\lambda}_l P_l\left(-i\frac{\partial}{\partial x}\right)\Delta(x, m). \quad (22)$$

The proof of the necessity of the condition can be given on the basis of the following theorem of Bogolyubov and Vladimirov[9] on the analytic continuation of generalized functions,† which we shall formulate suitably for the application to our case.

Let there be given two generalized functions $f_R(x)$ and $f_A(x)$, which vanish in the respective regions

$$x_0 < 0 \text{ or } x^2 < 0, \qquad x_0 > 0 \text{ or } x^2 < 0. \quad (23)$$

Let their Fourier transforms $f_{R,A}(k)$ coincide in the region

$$k^2 < m^2. \quad (24)$$

Then one can find a positive integer n such that in

*From considerations of invariance, $P_l(k)$ can be represented as a polynomial in the various invariants that can be constructed from k and the other four-vectors that occur in $\tilde{\lambda}(k)$.

†I take occasion to express my sincere gratitude to V. S. Vladimirov, who called my attention to the proof given here.

the region (24) these functions can be represented in the form

$$f_R(k) = f_A(k) = \sum_{l=0}^{n} P_l(k)\, \Phi_l(k^2)\ \ldots, \quad (25)$$

where $P_l(k)$ are polynomials, and the functions $\Phi_l(k^2)$ admit of analytic continuation to the entire plane of the complex variable z except the cut*

$$\operatorname{Im} z = 0, \qquad \operatorname{Re} z \geqslant m^2. \quad (26)$$

The important points for us in this theorem are, first, that the $\Phi_l(k^2)$ depend on k only through the invariant k^2, and second, that from the analytic character of $\Phi_l(z)$ there follows the representation

$$\Phi_l(z) = \int_{m^2}^{\infty} \frac{\rho_l(\zeta)\, d\zeta}{\zeta - z}. \quad (27)$$

Furthermore,

$$f_{R,A}(k) = \lim \sum_{l=0}^{n} P_l(k)\, \Phi_l(z)$$

$$\text{for } z \to k^2 \pm ik_0\varepsilon; \qquad \varepsilon > 0 \quad (28)$$

and, consequently,

$$f_R(k) - f_A(k) = 2\pi i \sum_{l=0}^{n} P_l(k)\, \varepsilon(k_0)\, \rho_l(k^2) \quad (29)$$

in the region $k^2 \geq m^2$.

If we now set

$$f_{R,A}(x) = \pm\, \theta(\pm x_0)\, f(x), \quad (30)$$

where f(x) is defined by (19), then these functions satisfy all of the conditions of the theorem. The conditions (23) are satisfied owing to the fact that $f(x) = 0$ for $x^2 < 0$. The Fourier transform $\tilde{f}(k)$ of the function $f(x) = f_R(x) - f_A(x)$ is of the form

$$\tilde{f}(k) = 2\pi i \varepsilon(k_0)\delta(k^2 - m^2)\, \tilde{\lambda}(k), \quad (31)$$

i.e., it clearly satisfies the condition (24). Now, comparing (29) and (31), we find

$$(2\pi i)^{-1} \tilde{f}(k) = \varepsilon(k_0)\, \delta(k^2 - m^2)\, \tilde{\lambda}(k) = \sum_{l=0}^{n} P_l(k)\, \rho_l(k^2)\, \varepsilon(k_0).$$

From this it follows that $\rho_l(k^2) = \delta(k^2 - m^2)\tilde{\lambda}_l$. This proves the necessity of the representation (21) for $\tilde{\lambda}(k)$ on the hyperboloid (20).†

It is easily verified that the necessary and sufficient restrictions on $\Lambda(x, y)$ in (15) which we have found, and which follow from the requirement that the commutator $[\varphi_{in}(x), j(y)]$ vanish outside the light cone, automatically assure also that the commutator (9c) is zero outside the light cone.

Let us point out two consequences of the conditions (15) and (21) for the local solubility of (8).

1. On the mass shell $\Lambda(x, y)$ behaves like a quasi-local operator. Let us consider

$$I \equiv \int \exp(-ikx)\, \langle p \,|\, \Lambda(x, y) \,|\, p' \rangle\, d^4x, \quad (32)$$

for $k^2 = m^2$. Transforming I by means of (18) and (21), we find

$$I = \exp[i(Q - k)y] \sum_{l=0}^{n} \tilde{\lambda}_l P_l(k). \quad (33)$$

On the other hand, we can get (33) from (32) if we set

$$\langle p \,|\, \Lambda(x, y) \,|\, p' \rangle = \exp(iQy) \sum_{l=0}^{n} \tilde{\lambda}_l P_l\left(-i\frac{\partial}{\partial x}\right) \delta(x - y). \quad (34)$$

2. Owing to the different natures of the terms $i\theta(x_0 - y_0)[j(x), j(y)]$ and $\Lambda(x, y)$ in the right member of (15) it is obvious that they cannot compensate each other. In particular this means that if $\delta j(x)/\delta\varphi_{in}(y) = 0$, or in other words if the operator j(x) does not depend on $\varphi_{in}(y)$, Eq. (8) has no local solutions for $\varphi(x)$. The examples considered by Wightman and Epstein[7] refer to just this class of operators j(x), and consequently do not satisfy the conditions for local solubility of (8).

3. THE EQUIVALENCE OF THE TWO APPROACHES ON THE MASS SHELL

Let us find the additional restrictions on the operator $\Lambda(x, y)$ in (15) that are required by the unitarity of the S matrix and the relation (7) between j(x) and S. Substitution of (15) in (11) gives

$$\Lambda(x, y) = \Lambda(y, x). \quad (35)$$

Taking the second variational derivative of $S^+S = 1$ with respect to $\varphi_{in}(x)$ and using (7) and (15), we get

$$\Lambda(x, y)^* = \Lambda(x, y). \quad (36)$$

Thus the necessary conditions for the local solubility of the equation (8) along with a unitary S matrix and the connection (7) between j(x) and S are (15), (21), (35), and (36).

Let us now compare the reduction formulas in the two methods. If we start from the causality conditions (6), then according to (11)

$$[a_{in}(\mathbf{q}), j(x)] = -i(2\pi)^{-3/2} \int \frac{e^{iqy} d^4y}{(2q_0)^{1/2}}\, \theta(x_0 - y_0)\, [j(x), j(y)], \quad (37)$$

where

*In what follows we shall assume for simplicity that the $\Phi_l(z)$ fall off for $|z| \to \infty$.

†We note that the representation (21) also follows from the integral representation of Jost, Lehmann, and Dyson.[10]

$$q_0 = +(\mathbf{q} + m^2)^{1/2}.$$

From the local solubility of (8) it follows that

$$[a_{in}(\mathbf{q}),\, j(x)] = (2\pi)^{-3/2}\int \frac{e^{iqy}\, d^4y}{(2q_0)^{1/2}}\{-i\theta(x_0 - y_0)[j(x), j(y)] + \Lambda(x, y)\}. \quad (38)$$

But when $q^2 = m^2$ [cf. Eqs. (32) — (34)] $\Lambda(x, y)$ in Eq. (38) behaves like a quasi-local operator, which we have not taken into account in (37). Therefore the reduction formulas (37) and (38) in the two methods give identical results. Similarly it can be shown that in both methods

$$\left[a_{in}(q),\, \frac{\delta S}{\delta\varphi_{in}(x)}\right] = -(2\pi)^{-3/2}\int \frac{e^{iqy}\, d^4y}{(2q_0)^{1/2}} ST(j(x), j(y)). \quad (39)$$

From this we quickly find that an arbitrary matrix element $\langle p_1, \ldots, p_n | S | q_1, \ldots, q_m \rangle$ can be put in the form $(p_i \neq q_j)$

$$(-i)^{m+n}(2\pi)^{-\frac{3(n+m)}{2}} \prod_{i=1}^{n}(2p_i^0)^{-1/2} \prod_{j=1}^{m}(2q_j^0)^{-1/2} \times \int \exp\left(\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} q_j y_j\right) \times \langle 0 | Tj(x_1) \ldots j(x_n)\, j(y_1) \ldots j(y_m) | 0 \rangle \prod_i d^4x_i \prod_j d^4y_j, \quad (40)$$

where we have dropped quasi-local terms.

We summarize briefly the results that have been obtained. First, the Bogolyubov causality condition (6) and the causality conditions in the form (9a) — (9c) together with Eq. (8) lead to identical expressions (40) for the matrix elements of the S matrix on the mass shell. Second, in solving this problem there is no need to use the asymptotic conditions in the form proposed by LSZ.[4] This is a sharpening of the initial postulates in the second method of extrapolation as compared with the original formulation of LSZ.[3,4] Third and last, it is important to note that the addition to $S_n(x_1, \ldots x_n) = (-i)^n \langle 0 | Tj(x_1) \ldots j(x_n) | 0 \rangle$ of terms $\Lambda_n(x_1, \ldots x_n)$ which behave like quasi-local terms on the mass shell (and owing to the unitarity of S have the property $\Lambda_n^* = \Lambda_n$ and are symmetric under interchange of any pair, $x_i \longleftrightarrow x_j$) does not destroy the local [in the sense of Eq. (9)] properties of the operators $\varphi(x)$ and $j(x)$ constructed from this S matrix by means of (7) and (8), but does contradict the causality condition in the form (6).

It is interesting to compare these results with those of other papers. In the papers of LSZ[3,4] the basic initial postulates include along with the causality conditions (9a) — (9c) the asymptotic conditions

$$\langle \Psi | a(\mathbf{q}, x_0) | \Phi \rangle \to \langle \Psi | a_{in,\,out}(\mathbf{q}) | \Phi \rangle, \qquad x_0 \to \mp\infty, \quad (41)$$

where $|\Psi\rangle$ and $|\Phi\rangle$ are arbitrary states and

$$a(\mathbf{q}, x_0) = (2\pi)^{-3/2}\int d\mathbf{x}\left\{f_{\mathbf{q}}^*(x)\frac{\partial\varphi(x)}{\partial x_0} - \varphi(x)\frac{\partial f_{\mathbf{q}}^*(x)}{\partial x_0}\right\},$$

$$f_{\mathbf{q}}(x) = (2q_0)^{-1/2}\exp(iqx), \qquad q_0 = +(\mathbf{q}^2 + m^2)^{1/2}.$$

The causality conditions (9a) — (9c) are never used explicity in references 3, 4, 11, and 12 to obtain covariant expressions for the matrix elements of S, nor to derive the various relations between the Green's functions, the multiple retarded commutators, and the expansions of $\varphi(x)$ and $j(x)$ in functional series in $\varphi_{in}(x)$ or $\varphi_{out}(x)$. This has created the illusion that the asymptotic conditions (41), without causality, are sufficient for the obtaining of these relations, and that in general such relations can also be correct in a nonlocal theory. Kaschluhn[5] was the first to show that the application of the asymptotic conditions in the papers of LSZ is not sufficiently well defined, and the expression for the matrix elements $\langle p | S | p' \rangle$ in the form (40) does not follow from the asymptotic conditions alone. If, however, one uses the asymptotic conditions as they are used in the papers of LSZ[3,4,11] then the operators $\varphi(x)$ of the interacting field must necessarily commute outside the light cone, in order that there be no contradictions with relativistic invariance.

These conclusions of Kaschluhn are fully confirmed by the results of the present paper. Furthermore, if we start from Eqs. (8) and (9a) — (9c), then, as has already been remarked, there is no need to use the asymptotic conditions (41). As for the possibility that there exist nonlocal solutions of (8), without resorting to Eqs. (9a) — (9c) or to Eq. (6), and to additional assumptions of the type of the adiabatic hypothesis, in our opinion the question remains an open one.* Whereas the role of the asymptotic conditions (41) in the derivation of Eq. (40) from Eqs. (8) — (9c) is not an essential one, the requirements of causality, Eqs. (9a) and (9b) are of decisive importance.

Here it is necessary to emphasize once more that the condition (9a) on one hand, and the conditions (9b) and (9c) on the other, are not equivalent. This result of the present paper contradicts Kaschluhn's conclusion[6] that "the commutation condition for the operators of the interacting field

*We note that if we start from Eq. (8), without assuming the connection (7) between j(x) and the S matrix, the possibility of the existence of nonlocal solutions can be settled in a trivial way (see the analysis given for the Wightman example). This possibility, it is true, is of no interest from the physical point of view.

cannot be interpreted as a condition on the reduced elements of the D matrix, from which follow some analytical consequences for the theory of dispersion relations" (cf. pages 4, 5, and 33 in reference 6). The mistake in this conclusion comes from the fact that in studying the various expressions involving operators $\varphi(x)$ of the interacting field Kaschluhn has actually nowhere taken into account the additional restrictions on the operators $\varphi(x)$ and $j(x)$ that arise from the conditions (9a) and (9b) for local behavior.

In connection with the equivalence of the two methods from the point of view of the properties of $S_n(x_1, \ldots x_n)$ in the S matrix (1) on the mass shell the question can arise: Where is the retardation condition imposed in the formation of causality in the form (9a) and (9b)? Essentially, it is contained in the expression of $\varphi(x)$ in terms of $j(x)$ in Eq. (8) by means of retarded (or advanced) Green's functions $\Delta^R(x, m)$ [or $\Delta^A(x, m)$ of the homogeneous Klein-Gordon equation. In this connection, however, it must hold apart from terms that vanish on the mass shell, i.e., it does not have to hold rigorously.

Comparing the two methods that have been considered, we note an important advantage of the Bogolyubov method, which is that the condition (6) is formulated and can be used without any resort to the operators $\varphi(x)$ of the interacting field and Eq. (8) for these operators. Moreover, for obtaining the expressions for $S_2(x_1, x_2)$ and $S_3(x_1, x_2, x_3)$ in Eq. (1), which correspond to the single-particle and vertex Green's functions, in the forms

$$S_2 = -\langle 0 | Tj(x_1), j(x_2) | 0 \rangle,$$
$$S_3 = -i \langle 0 | Tj(x_1), j(x_2), j(x_3) | 0 \rangle$$

and for studying their analytical properties, the causality conditions (6) are not only sufficient, but also necessary conditions. This is so because the contributions from $S_2(x_1, x_2)$ and $S_3(x_1, x_2, x_3)$ in Eq. (1) drop out on the mass shell. Therefore one cannot get unambiguous expressions for these functions by means of the conditions (9a) and (9b).

On the other hand, all of the difficulties of present quantum field theory in its Lagrangian formulation have their roots, as a rule, in the properties of just these simplest Green's functions. Therefore the existence of an additional arbitrariness in their definition, which does not contradict the local character in the sense of Eq. (9), may be due to deep causes. In any case the requirement (9) is weaker than Eq. (6), and in general widens the range of possibilities.

In conclusion we point out an interesting consequence of the theorem proved in Sec. 2: the commutators $[\varphi_{in,out}(x), j(y)]$, $[\varphi_{in,out}(x), \varphi(y)]$, and $[\varphi_{in}(x), \varphi_{out}(y)]$ do not vanish outside the light cone,* $(x-y)^2 < 0$, if we exclude the trivial case in which $\varphi_{in}(x) = \varphi_{out}(x)$ or $\varphi(x)$ and $j(x)$ do not depend on $\varphi_{in}(x)$ [or $\varphi_{out}(x)$]. For example,

$$[\varphi_{in}(x), j(y)] = \int \Delta(x-x', m)\, \theta(y_0 - x'_0)\, [j(x'), j(y)]\, d^4x' \neq 0 \text{ for } x \sim y$$

since otherwise it would be necessary for an arbitrary matrix element $\langle p | [j(x), j(y)] | p' \rangle$ to behave on the mass shell like a quasi-local operator [$\sim \delta(x-y)$ and its covariant derivative].

The method used in the present paper can be extended without particular difficulty to the case of several interacting fields and the presence of bound states.

[1] N. N. Bogolyubov, Izv. Akad. Nauk SSSR, Ser. Fiz. **19**, 237 (1955), Columbia Tech. Transl. p. 215.

[2] N. N. Bogolyubov and D. V. Shirkov, Introduction to Quantized Field Theory, Interscience, 1959. Bogolyubov, Medvedev, and Polivanov, Вопросы теории дисперсионных соотношений (Problems in the Theory of Dispersion Relations), Fizmatgiz, 1958.

[3] Lehmann, Symanzik, and Zimmermann, Nuovo cimento **1**, 205 (1955).

[4] Lehmann, Symanzik, and Zimmermann, Nuovo cimento **6**, 319 (1957).

[5] F. Kaschluhn, Nuovo cimento **12**, 541 (1959).

[6] F. Kaschluhn, On the Asymptotic and Causality Conditions in Quantum Field Theory, II, Preprint r-517, Dubna, 1960.

[7] A. Wightman, Report at the International Conference on High-Energy Physics, Kiev, 1959. A. Wightman and H. Epstein, Ann. of Phys. **11**, 201 (1960).

[8] R. Haag, Nuovo cimento suppl. **14**, 131 (1959); Kgl. Dansk. Vidensk. Selskab, mat.-fys. medd. **29**, No. 12 (1955). D. Hall and A. S. Wightman, Kgl. Dansk. Vidensk. Selskab, mat.-fys. medd. **31**, No. 5 (1957). O. W. Greenberg, Phys. Rev. **115**, 706 (1959).

*In this case Miyatake's assertion[13] that vanishing of the commutator $[\varphi_{in}(x), \varphi_{out}(y)]$ outside the light cone is a condition for causality in a local theory seems strange, to say the least.

[9] N. N. Bogolyubov and V. S. Vladimirov, Научн. докл. высш. шк. (Sci. Rep. of the Universities, Phys. Math. Sci.) **3**, 26 (1958).

[10] R. Jost and H. Lehmann, Nuovo cimento **5**, 1598 (1957). F. J. Dyson, Phys. Rev. **110**, 1460 (1958).

[11] Glaser, Lehmann, and Zimmermann, Nuovo cimento **6**, 1122 (1957).

[12] K. Nishijima, Progr. Theor. Phys. **17**, 765 (1957); Phys. Rev. **119**, 485 (1960).

[13] Y. Miyatake, Progr. Theor. Phys. **21**, 562 (1959).

Translated by W. H. Furry
300