STABILITY OF MAGNETIC TANGENTIAL DISCONTINUITIES IN RELATIVISTIC HYDRODYNAMICS

M. T. ZHUMARTBAEV

Institute of Nuclear Physics, Academy of Sciences, Kazakh S.S.R.

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Small perturbations of the discontinuity surface in relativistic magnetohydrodynamics are examined. It is shown that in the case of small perturbations the magnetic tangential discontinuity is conserved as such. Stability of magnetic tangential discontinuities is investigated. In particular, their instability region is determined in the ultrarelativistic case.

1. INTRODUCTION

THE stability of magnetic tangential discontinuities in nonrelativistic hydrodynamics was investigated by Syrovat-skii.[1,2] He came to the important conclusion that the magnetic field stabilizes the flow. In this connection, great interest is attached to investigations of the stability of a magnetic tangential discontinuity in relativistic hydrodynamics. In the first part of the present article we derive the boundary conditions for a perturbed discontinuity on the basis of the general Lorentz transformation.* It follows from these conditions that the magnetic tangential discontinuity is conserved as such under small perturbations.

In the second part we investigate the stability of a magnetic tangential discontinuity. Conditions are derived for the stability in two particular cases, of large and small angles between the wave vector $\mathbf{k}_0$ and the direction common to the parallel vectors **v** and **H**. All the calculations were made for a medium with infinite conductivity, and in the nonrelativistic approximation the final formulas go into those obtained by Syrovat-skii.[1,2]

2. PERTURBATION OF DISCONTINUITY SURFACE

In the coordinate system tied to the unperturbed discontinuity, the following conservation laws apply on the normal to the discontinuity surface:

$$\{Wu_f^2 + p + (H^2 - 2H_f^2 + E^2 - 2E_f^2)/8\pi\} = 0, \quad (2.1)$$

$$\{Wu_f u_\tau - (H_f H_\tau + E_f E_\tau)/4\pi\} = 0, \quad (2.2)$$

$$\{Wu_f u_4 + i([\mathbf{EH}]\,\mathbf{f})/4\pi\} = 0, \quad (2.3)\dagger$$

$$\{nu_f\} = 0, \quad (2.4)$$

$$\{H_f\} = 0, \quad (2.5)$$

$$\{\mathbf{E}\boldsymbol{\tau}\} = 0 \quad (2.6)$$

where **f** is a unit vector normal to the discontinuity surface, τ the direction of the tangent, and W the heat function per unit volume; the braces denote the difference between the corresponding values on the two sides of the discontinuity.

We now change to a coordinate system in which the normal velocity of the perturbed discontinuity is zero. We make the change with the aid of the general Lorentz formula for the velocity and the field. The normal **f** and the two independent directions τ_1 and τ_2 tangent to the perturbed discontinuity surface have the following components

$$\mathbf{f}(1, -\partial\xi/\partial y, -\partial\xi/\partial z), \qquad \boldsymbol{\tau}_1(\partial\xi/\partial y, 1, 0),$$
$$\boldsymbol{\tau}_2(\partial\xi/\partial z, 0, 1).$$

We can then readily obtain in the new system of coordinates, the boundary conditions on the perturbed discontinuity accurate to terms up to second order of smallness relative to the perturbations of the velocity $\delta\mathbf{v}$ and of the field **h** (in the case of infinite conductivity), using Eqs. (2.1) – (2.6) for the parameters of the unperturbed discontinuity.

After simple transformations, introducing the notation

$$h_f = h_x - H_y\partial\xi/\partial y - H_z\partial\xi/\partial z, \qquad \gamma = \sqrt{1 - v^2/c^2},$$
$$v_f + \delta v_f = \delta v_x - \partial\xi/\partial t - v_y\partial\xi/\partial y - v_z\partial\xi/\partial z, \quad (2.7)$$

we obtain the following boundary conditions for the tangential discontinuity ($v_x = 0$, $H_x = 0$):

$$\left\{\delta p + \frac{1}{4\pi}\left[(\mathbf{Hh}) + \frac{1}{c^2}[\mathbf{vH}]_x([\mathbf{H}\delta\mathbf{v}]_x - [\mathbf{vh}]_x)\right]\right\} = 0, \quad (2.8)*$$

*The need for using the general Lorentz transformation in the solution of three-dimensional problem is mentioned by Stanyukovich.[3] The same reference considers several general transformations for a medium with variable μ and ε.

†$[\mathbf{E\,H}] = \mathbf{E}\times\mathbf{H}$.

*$(\mathbf{H\,h}) = \mathbf{H}\cdot\mathbf{h}$.

$$\left\{\frac{w}{\gamma}+\frac{H^2}{4\pi n}\gamma\right\}\frac{n(v_f+\delta v_f)}{\gamma}=\frac{1}{4\pi}\{(\mathbf{vH})\}h_f, \qquad (2.9)$$

$$\frac{1}{4\pi}\left\{H_y+\frac{v_z}{c^2}[\mathbf{vH}]_x\right\}h_f=\left\{\frac{wv_y}{c^2\gamma}+\frac{[\mathbf{vH}]_x}{4\pi nc^2}\gamma\right\}\frac{n(v_f+\delta v_f)}{\gamma}, \qquad (2.10)$$

$$\frac{1}{4\pi}\left\{H_z-\frac{v_y}{c^2}[\mathbf{vH}]_x\right\}h_f=\left\{\frac{wv_z}{c^2\gamma}-\frac{[\mathbf{vH}]_x}{4\pi nc^2}\gamma\right\}\frac{n(v_f+\delta v_f)}{\gamma}, \qquad (2.11)$$

$$\{v\ \}h_f=\left\{\frac{1}{n}H_y\gamma\right\}\frac{n(v_f+\delta v_f)}{\gamma}, \qquad (2.12)$$

$$\{v_z\}h_f=\left\{\frac{1}{n}H_z\gamma\right\}\frac{n(v_f+\delta v_f)}{\gamma}, \qquad (2.13)$$

where w is the heat function per particle.

The system of five equations (2.9) — (2.13) for the two unknowns h_f and $n(v_f+\delta v_f)/\gamma$ admits of only one vanishing trivial solution

$$h_f=h_x-H_y\partial\xi/\partial y-H_z\partial\xi/\partial z=0,$$
$$v_f+\delta v_f=\delta v_x-\partial\xi/\partial t-v_y\partial\xi/\partial y-v_z\partial\xi/\partial z=0. \qquad (2.14)$$

Thus, the magnetic tangential discontinuity is conserved as such under small perturbations.

3. STABILITY OF MAGNETIC TANGENTIAL DISCONTINUITY

For perturbations of the form $\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)$, we can readily reduce the system $\partial T_i^k/\partial x^k=0$, using Maxwell's and the continuity equations, to a system of equations in three unknowns δv_x, δv_y, and δv_z. The required existence of a nontrivial solution of such a system leads to a dispersion equation of the form

$$a_1\omega_0^6+a_2\omega_0^5+a_3\omega_0^4+a_4\omega_0^3+a_5\omega_0^2+a_6\omega_0+a_7=0, \qquad (3.1)$$

where $\omega_0=\mathbf{k}\cdot\mathbf{v}-\omega$, and the coefficients are algebraic functions of W, the velocity of sound c_0, k^2, $(\mathbf{k}\cdot\mathbf{v})$, etc.

We can separate from (3.1) the factor

$$F=\left[1+\gamma^2\frac{H^2}{4\pi W}\right]\frac{\omega_0^2}{\gamma^2}-\frac{(\mathbf{vH})(\mathbf{kH})}{2\pi W}\omega_0-\frac{c^2(\mathbf{kH})^2}{4\pi W}\gamma^2. \qquad (3.2)$$

The equation $F=0$ yields a solution, known to be stable, with real ω, and can be discarded.

Separation of the second factor yields

$$\frac{1}{\gamma^4}\left\{\left[1+\gamma^2\frac{H^2}{4\pi W}\right]+\frac{(1-c_0^2/c^2)}{c^2\gamma^2}\left[v^2+\gamma^2\frac{(\mathbf{vH})^2}{4\pi W}\right]\right\}\omega_0^4$$
$$-\frac{1}{\gamma^4}\left\{2\frac{c_0^2}{c^2}\left[\mathbf{kv}+\gamma^2\frac{(\mathbf{vH})(\mathbf{kH})}{4\pi W}\right]+\frac{\mathbf{kv}}{2\pi W}\left[\gamma^2H^2\right.\right.$$
$$\left.\left.+\left(1-\frac{c_0^2}{c^2}\right)\frac{(\mathbf{vH})^2}{c^2}\right]\right\}\omega_0^3-\frac{1}{\gamma^2}\left\{\left\langle\frac{c^2}{4\pi W}\left[H^2+\frac{(1-c_0^2/c^2)}{c^2\gamma^2}(\mathbf{vH})^2\right]\right.\right.$$
$$\left.\left.+\frac{c_0^2}{\gamma^2}\right\rangle\left[k^2-\frac{1}{c^2}(\mathbf{kv})^2\right]+\frac{c_0^2\mathbf{kH}}{4\pi W}\left[\gamma^2\mathbf{kH}-4\frac{(\mathbf{kv})(\mathbf{vH})}{c^2}\right]\right\}\omega_0^2$$
$$+\frac{c_0^2\mathbf{kH}}{2\pi W\gamma^2}\left\{\gamma^2(\mathbf{kH})(\mathbf{kv})+\mathbf{vH}\left[k^2-\frac{1}{c^2}(\mathbf{kv})^2\right]\right\}\omega_0$$
$$+\frac{c_0^2c^2}{4\pi W}(\mathbf{kH})^2\left[k^2-\frac{1}{c^2}(\mathbf{kv})^2\right]=0. \qquad (3.3)$$

Instability corresponds to values of k_x whose imaginary part differs from zero. We put $k_x=-i$ and then the perturbation will be bounded away from the discontinuity, provided the conditions

$$\mathrm{Re}\,\lambda_1>0, \qquad \mathrm{Re}\,\lambda_2<0, \qquad (3.4)$$

are satisfied, where λ_1 and λ_2 pertain to the regions $x<0$ and $x>0$ respectively. By means of Eq. (3.3) we can express the dependence of λ on k:

$$\lambda=k_0\sqrt{1-k^2/k_0^2}. \qquad (3.5)$$

As shown in the preceding section, the boundary conditions (2.14) should hold on the perturbed surface of the tangential discontinuity:

$$\delta v_x-i\omega_0\xi=0, \qquad h_x-i\mathbf{k}\mathbf{H}\xi=0. \qquad (3.6)$$

We must also add to these conditions the boundary equation (2.8). Eliminating ξ from (3.6), we obtain the independent boundary condition

$$h_{x_1}/\mathbf{k}\mathbf{H}_1=h_{x_2}/\mathbf{k}\mathbf{H}_2. \qquad (3.7)$$

From the system of equations of relativistic magnetohydrodynamics we can readily derive the following expression for small velocity and magnetic-field perturbations:

$$\delta\mathbf{v}=-c_0^2\frac{\delta n}{\omega_0 n\Delta_0}(A\mathbf{k}+B\mathbf{v}+C\mathbf{H}), \qquad (3.8)$$

$$\mathbf{h}=-c_0^2\frac{\delta n}{\omega_0^2 n\Delta_0}\{\mathbf{kH}(A\mathbf{k}+B\mathbf{v})-(Ak^2+B\mathbf{kv})\mathbf{H}\}, \qquad (3.9)$$

where $A=\omega_0^2/\gamma^4$; $B=\omega_0^2(\omega_0-\mathbf{k}\cdot\mathbf{v})/c^2\gamma^4$, C and Δ_0 are functions of ω_0, H, W, k, and v.

Eliminating with the aid of (3.7) — (3.9) the components $\delta\mathbf{v}$ and $\mathbf{h}$ from the boundary equation (2.8), and using (3.3), we obtain after several transformations an equation that determines the possible values of ω:

$$\lambda_1/W_1F_1=\lambda_2/W_2F_2, \qquad (3.10)$$

where λ is expressed in terms of k^2 by means of (3.5), and F is defined in (3.2).

An investigation of the stability of a magnetic tangential discontinuity in relativistic hydrodynamics reduces to an investigation of the roots of (3.10). If for certain values of the parameters contained in (3.10) the equation has no root ω with positive imaginary parts (for all values of k_0), then this magnetic tangential discontinuity is stable against the small perturbations considered here. In the opposite case, it is unstable.

If there is no magnetic field (i.e., $H_1=0$, $H_2=0$), we arrive at the less interesting hydrodynamic problem, that of stability of a tangential discontinuity. It is known that such a discontinuity is absolutely unstable. To investigate the roots of (3.2) we confine ourselves, following Syrovat-skii,[2] to the particular case of discontinuities, in which the flow of the medium is along the magnetic field, i.e., when the vectors $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are parallel in some system of coordinates.

We denote the cosine of the angle between the direction of the wave vector $\mathbf{k}_0$ and the general direction of the vectors $\mathbf{v}$ and $\mathbf{H}$ by ν. We first investigate the roots of (3.10) as $\nu \to 0$. The equation (3.10) is of the same order in ω as the corresponding equation in Syrovat-skii's paper,[2] because it has one multiple root $\omega = 0$ when $\nu = 0$, and two simple real roots. But since the transition from the real root to the imaginary one (to the pair of complex conjugate roots) is possible only through a multiple root, the imaginary roots of (3.10) should tend to zero together with ν. This enables us to neglect the ratio k^2/k_0^2 under the radical sign as much smaller than unity, and then ω is given by

$$W_1\left\{\frac{1}{\gamma_1^2}\left[1+\gamma_1^2\frac{H_1^2}{4\pi W_1}\right]\left(\frac{\omega}{k_0}-v_1\nu\right)^2+\frac{v_1H_1^2\nu}{2\pi W_1}\left(\frac{\omega}{k_0}-v_1\nu\right)-\frac{c^2H_1^2\nu^2}{4\pi W_1}\gamma_1^2\right\}=-W_2\left\{\frac{1}{\gamma_2^2}\left[1+\gamma_2^2\frac{H_2^2}{4\pi W_2}\right]\left(\frac{\omega}{k_0}-v_2\nu\right)^2+\frac{v_2H_2^2\nu}{2\pi W_2}\left(\frac{\omega}{k_0}-v_2\nu\right)-\frac{c^2H_2^2\nu^2}{4\pi W_2}\gamma_2^2\right\}. \quad (3.11)$$

The roots of (3.11) are real if the following condition is satisfied

$$\left(\frac{H_1^2}{4\pi}+\frac{H_2^2}{4\pi}\right)+\frac{(H_1^2/4\pi+H_2^2/4\pi)^2}{W_1+W_2}-\frac{W_1W_2}{c^2(W_1+W_2)}\frac{(v_2-v_1)^2}{\gamma_1^2\gamma_2^2}\geqslant 0. \quad (3.12)$$

The second term of this condition tends to zero in the non-relativistic limit, but at relativistic energy density it can make a considerable contribution to the stabilization of the flow.

An investigation of the roots of (3.10) for the case of arbitrary ν is best carried out for a narrower class of discontinuities, namely when the heat function, the velocity of sound, and the field intensity have the same values on both sides of the discontinuity:

$$W_1 = W_2 \equiv W, \qquad c_{01} = c_{02} \equiv c_0, \qquad H_1 = H_2 \equiv H. \quad (3.13)$$

If $v_0 = (v_2 - v_1)/(1 - v_1v_2/c^2)$ ($\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel) is the value of the jump on the discontinuity, then in the "symmetrical" system of coordinates we have

$$v_1 = -\frac{v_0}{2}\left(1-\frac{v_1v_2}{c^2}\right), \qquad v_2 = \frac{v_0}{2}\left(1-\frac{v_1v_2}{c^2}\right).$$

Solving these equations for v_1 and v_2, we find

$$v_1 = -\frac{c^2}{v_0}\left(1-\sqrt{1-\frac{v_0^2}{c^2}}\right) = -V_0,$$
$$v_2 = \frac{c^2}{v_0}\left(1-\sqrt{1-\frac{v_0^2}{c^2}}\right) = V_0. \quad (3.14)$$

Introducing the notation

$$\Omega = \omega/c_0k_0, \qquad \alpha = H/\sqrt{4\pi W},$$
$$\beta_0 = V_0/c_0, \qquad \beta = v_0/c_0, \quad (3.15)$$

squaring both halves of (3.10), and discarding the root $\Omega = 0$, we obtain an algebraic equation of the eighth degree for Ω, in the form

$$b_1\Omega^8 + b_2\Omega^6 + b_3\Omega^4 + b_4\Omega^2 + b_5 = 0, \quad (3.16)$$

where the coefficients are functions of the parameters α, β_0, $y = c_0/c$, and ν. Ω should vanish on the boundary of the instability region.

At certain values of the parameters, the free term b_5 may vanish, and then (3.16) has a multiple root $\Omega^2 = 0$. If b_5 reverses sign as it goes through zero, Ω^2 also reverses sign, and this effects a transition from a pair of real roots to a pair of complex-conjugate roots (pure imaginary). Then the equation $b_5 = 0$ specifies, for a given ν, a curve in the plane (α, β_0), which is the region of the boundary of the instability region.

The equation $b_5 = 0$ yields two curves:

$$\alpha^2 = y^2\beta_0^2/(1-y^2\beta_0^2), \quad (3.17)$$

$$(\alpha/y)^4\{2(1-\beta_0^2)^2+\nu^2\beta_0^2(1-y^2)(2-y^2\beta_0^2-\beta_0^2)\} - (\alpha/y)^2\beta_0^2\{4(1-\beta_0^2)^2+\nu^2\beta_0^2(1-y^2)(1-2y^2\beta_0^2+\beta_0^2)/(1-y^2\beta_0^2)\}+2\beta_0^4-\nu^2\beta_0^6(1-y^2)/(1-y^2\beta_0^2)=0. \quad (3.18)$$

Equation (3.18) defines a real curve only when

$$\nu^2 \geqslant \frac{8[1-y^2(1-y^2\beta_0^2+\beta_0^2)]}{8+(1-\beta_0^2)^2-8y^2\beta_0^2(2-y^2\beta_0^2)}\,\frac{1-y^2\beta_0^2}{1-y^2}. \quad (3.19)$$

The condition (3.19) yields two intervals of values of β_0 for each value of ν:

$$\beta_0^2 \leqslant \frac{\nu^2+8y^2(\nu^2-1)-2\nu^2(1-y^2)\sqrt{2(\nu^{-2}-1)}}{\nu^2+8y^4(\nu^2-1)}, \quad (3.20)$$

$$\beta_0^2 \geqslant \frac{\nu^2+8y^2(\nu^2-1)+2\nu^2(1-y^2)\sqrt{2(\nu^{-2}-1)}}{\nu^2+8y^4(\nu^2-1)}. \quad (3.21)$$

The intervals (3.20) and (3.21) are meaningful only when ν varies within the limits

$$0 \leqslant \nu \leqslant 1.$$

We shall henceforth consider the ultrarelativistic case, i.e., $y = 1/\sqrt{3}$. In this case, according to (3.15), β and β_0 change in the same interval of values from 0 to $\sqrt{3}$. When $\nu^2 = 1$, (3.20) and (3.21) imply $\beta_0^2 \leq 1$ and $\beta_0^2 \geq 1$, respectively.

As $\nu \to 0$ we obtain $\beta_0 \to \sqrt{3}$, and curve (3.17) is the boundary of the stability region. Equation (3.17) coincides with one of the real roots of (3.12). In fact, in a "symmetrical" system of coordinates, the equation of the curve that serves as the boundary of the stability region at low values of ν, assumes the form

$$\alpha^4 + \alpha^2 - y^2\beta_0^2/(1 - y^2\beta_0^2)^2 = 0. \quad (3.22)$$

This equation has two roots:

$$\alpha_1^2 = y^2\beta_0^2/(1 - y^2\beta_0^2), \qquad \alpha_2^2 = -1/(1 - y^2\beta_0^2). \quad (3.23)$$

The second root of (3.23) has no physical meaning and should be discarded. Thus, as $\nu \to 0$ we arrive at the previous boundary (3.17) of the instability region. On the other hand, in the region $0 \leq \beta_0 \leq 1$ ($\nu = 1$) the limit is the curve (3.18). In fact, when $\nu = 1$, the equation $b_5 = 0$ assumes the form

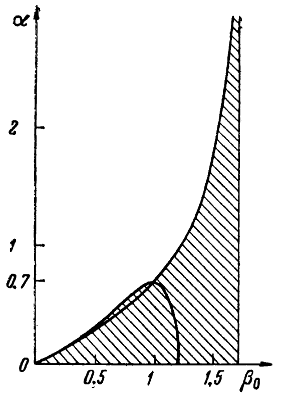
$$\left\{\alpha^2 - \frac{y^2\beta_0^2}{1 - y^2\beta_0^2}\right\}^2 \left\{\beta_0^4(1 - y^2)\left[1 + \frac{(1 - y^2)}{y^2}\alpha^2\right]\right.$$
$$\left. -2(1 - y^2\beta_0^2)\left[(\beta_0^2 - 1)\frac{\alpha^2}{y^2} + \beta_0^2\right]\right\} = 0. \quad (3.24)$$

It follows therefore that the vanishing of Ω^2 on the curve (3.17) occurs without a reversal in the sign, and the limit of the stability region when $\nu = 1$ is the curve

$$\alpha = \beta_0 \left[(3 - 2\beta_0^2)/(9 - 12\beta_0^2 + 5\beta_0^4)\right]^{1/2} \quad (3.25)$$

(we have put here $y = 1/\sqrt{3}$).

Thus, the region of instability of magnetic tangential discontinuities (in the ultrarelativistic case) is bounded on the left by the curve (3.25) when $\beta_0 \leq 1$ and by the curve (3.17) when $\beta_0 \geq 1$. The curves (3.17) and (3.25) are shown in the figure and the plane (α, β_0). The instability region is shaded. The discontinuities whose parameters lie outside the region will be stable against small perturbations.



We note that the stability criterion for a magnetic tangential discontinuity can be represented in the form

$$\frac{H^2}{8\pi} \geqslant \frac{1}{4}\left(\frac{1}{\sqrt{1 - v_0^2/c^2}} - 1\right)W, \quad (3.26)$$

where the expression in the right half (without the $1/4$) is the analog of the kinetic energy of a particle in relativistic hydrodynamics.

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[1] S. I. Syrovat-skii, JETP **24**, 632 (1953).

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