A KINETIC EXAMINATION OF SOME EQUILIBRIUM PLASMA CONFIGURATIONS

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Certain one-dimensional plasma configurations with effective dimensions of the order of the Larmor radius are analyzed.

INTRODUCTION

THERE are many plasma systems of practical interest whose properties cannot be described within the framework of magnetohydrodynamics. Typical examples are systems with acute-angle geometries and adiabatic traps with ion injection (Ogra, Astron). Problems involved in systems of this kind were first considered by Ferraro[1] in connection with the interaction of solar corpuscular streams with the magnetic field of the earth. Later, kinetic analyses of certain particular equilibrium configurations were carried out by a number of authors.[2-5] In the present work we develop a general approach to the problem and consider several concrete systems.

As is well known, magnetohydrodynamics applies when the Larmor radii of both electrons and ions are small compared with the scale of the field inhomogeneities. We, however, shall be interested in systems with dimensions of the order of the effective Larmor radius for electrons or ions. Here we can consider two limiting cases: in one there is a transition region of the order of the Larmor radius between the plasma (with no magnetic field) and the magnetic field; in the other, the entire region occupied by the plasma is characterized by dimensions of the order of the Larmor radius (Ogra, Astron).

In the present work we limit ourselves to one-dimensional problems (all quantities depend on one coordinate only).

1. GENERAL RELATIONS

We first consider the case in which all quantities depend only on the single Cartesian coordinate x. If collisions are neglected, the equilibrium plasma configuration is described by the Vlasov equations:

$$v_x \frac{\partial f_e}{\partial x} - \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)\frac{\partial f_e}{\partial \mathbf{v}} = 0, \qquad \text{(1a)*}$$

$$v_x \frac{\partial f_i}{\partial x} + \frac{e}{M}\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)\frac{\partial f_i}{\partial \mathbf{v}} = 0;$$

$$\operatorname{div}\mathbf{E} = 4\pi e\int (f_i - f_e)\,d\mathbf{v}, \qquad \operatorname{rot}\mathbf{H} = \frac{4\pi e}{c}\int \mathbf{v}(f_i - f_e)\,d\mathbf{v},$$

$$\mathbf{E} = -\nabla\Phi, \qquad \mathbf{H} = \operatorname{rot}\mathbf{A}. \qquad \text{(1b)*}$$

When the magnetic field does not have a longitudinal component H_x, the equations for the characteristics of (1a)

$$mv_x d\mathbf{v}/dx = -e(\mathbf{E} + c^{-1}[\mathbf{vH}]),$$

$$Mv_x\,d\mathbf{v}/dx = e(\mathbf{E} + c^{-1}[\mathbf{vH}]) \qquad (2)$$

have the six integrals

$$v_x^2 + v_y^2 + v_z^2 - 2e\Phi/m = C_{0e}^2, \qquad v_y - eA_y/mc = C_{ye},$$

$$v_z - eA_z/mc = C_{ze}, \qquad v_x^2 + v_y^2 + v_z^2 + 2e\Phi/M = C_{0i}^2,$$

$$v_y + eA_y/Mc = C_{yi}, \qquad v_z + eA_z/Mc = C_{zi}. \qquad (3)$$

However, if $H_x \neq 0$, the system in (2) has only four integrals. We shall limit ourselves to the case in which $H_x = 0$.

Knowing the complete set of integrals (3) we can write the general solution of the system (1a):

$$f_e = f_e(v^2 - 2e\Phi/m,\ v_y - eA_y/mc,\ v_z - eA_z/mc),$$

$$f_i = f_i(v^2 + 2e\Phi/M,\ v_y + eA_y/Mc,\ v_z + eA_z/Mc). \qquad (4)$$

Substituting Eq. (4) in Eq. (1b) we have

$$\frac{d^2\Phi}{dx^2} = -4\pi e\int (f_i(\mathbf{v}, \mathbf{A}, \Phi) - f_e(\mathbf{v}, \mathbf{A}, \Phi))\,d\mathbf{v},$$

$$\frac{d^2\mathbf{A}}{dx^2} = -\frac{4\pi e}{c}\int \mathbf{v}(f_i(\mathbf{v}, \mathbf{A}, \Phi) - f_e(\mathbf{v}, \mathbf{A}, \Phi))\,d\mathbf{v}. \qquad (5)$$

If f_i and f_e are given as functions of C_0^2, C_y, and C_z, then (5) determines the dependence of Φ and **A** on x. This system of equation has one first integral, which expresses the conservation of momentum:

$$\frac{H^2 - E^2}{8\pi} = \int v_x^2 (Mf_i + mf_e)\,d\mathbf{v}. \qquad (6)$$

The functions $f_{i,e}(C_0^2, C_y, C_z)$ are arbitrary functions of their arguments. In actual calculation they must be chosen on the basis of various phys-

*[vH] = v × H.

*rot = curl.

ical considerations, for example, stability considerations. In what follows we limit ourselves to configurations which may be called "single-Larmor" systems. In such systems, in which, for example, all the ions cross some plane $x = x_0$, the electrons may also have this property (two-component systems) or may be characterized by a Boltzmann distribution (single-component systems). A unique feature of such systems is the fact that assigning the distribution function for the "non-Boltzmann" particles in one plane $x = x_0$ is sufficient to determine the distribution of particles over the entire space. This situation is typical of a system in which injection is used.

Assuming that the potentials Φ and $\mathbf{A}$ vanish at $x = x_0$, we have

$$C_0^2 = v_0^2, \qquad C_y = v_{0y}, \qquad C_z = v_{0z}. \tag{7}$$

Whence we obtain the condition

$$C_0^2 \geqslant C_y^2 + C_z^2. \tag{8}$$

This condition determines the region of integration over $d\mathbf{v}$ in the right-hand side of the equations in (5). Substituting the expressions for C_0^2, C_y, and C_z from Eq. (3) in Eq. (8), we obtain the following condition on the region of integration over velocity:

$$v_x^2 + 2e\Phi/M \geqslant e^2A^2/M^2c^2 + 2e\mathbf{v}\mathbf{A}/Mc \tag{9*}$$

for the ions; a similar procedure holds for the electrons.

If the system has axial symmetry, all quantities depend only on r, and in this case $H_r = 0$; hence the equations for the characteristics of the system (1a) have a complete set of integrals

$$v^2 - 2e\Phi/m = C_{0e}^2, \qquad r\,(v_\varphi - eA_\varphi/mc) = C_{\varphi e},$$
$$v_z - eA_z/mc = C_{ze}, \qquad v^2 + 2e\Phi/M = C_{0i}^2,$$
$$r\,(v_\varphi + eA_\varphi/Mc) = C_{\varphi i}, \qquad v_z + eA_z/Mc = C_{zi}, \tag{10}$$

while (5) is replaced by

$$\frac{1}{r}\frac{d}{dr}\,r\frac{d\Phi}{dr} = -4\pi e\int (f_i - f_e)\,d\mathbf{v},$$
$$\frac{d}{dr}\frac{1}{r}\frac{d}{dr}\,rA_\varphi = -\frac{4\pi e}{c}\int v_\varphi\,(f_i - f_e)\,d\mathbf{v},$$
$$\frac{1}{r}\frac{d}{dr}\,r\frac{dA_z}{dr} = -\frac{4\pi e}{c}\int v_z\,(f_i - f_e)\,d\mathbf{v}. \tag{11}$$

The solution of (5) and (11) can be simplified considerably if the plasma is assumed to be neutral. In this case, from the fact that $n_i = n_e$ we obtain a relation between Φ and $\mathbf{A}$ and it is sufficient to consider the equation for $\mathbf{A}$ only.

2. PLANE ONE-COMPONENT SYSTEMS

As we have noted above, by one-component systems we mean systems in which the current is due

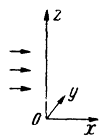

FIG. 1

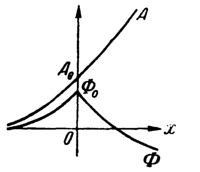

FIG. 2

to only one kind of particle; the second serves simply for charge compensation and is characterized by a Boltzmann distribution $n = n_0 \exp(-Ze\Phi/T)$. We shall see below that in general the mean energy of the "Boltzmann" particles must be much smaller than the energy of the "current-carrying" particles if the effect of the electric field on the motion of the latter is to be neglected.

A. Consider a monoenergetic ion beam normally incident on a magnetic field. Suppose that at $x = -\infty$ all the ions have the came velocity v_0, which is in the x direction (Fig. 1), while there are no magnetic or electric fields ($\Phi = 0$, $\mathbf{A} = 0$). At $x = +\infty$ the magnetic field is uniform along the axis and given by H_0, while the electric field, and the ion and electron densities all vanish.

Substituting in Eq. (5) the ion distribution function

$$f_i = 2v_0 n_0 \delta(v^2 + 2e\Phi/M - v_0^2)\,\delta(v_y + eA/Mc)\,\delta(v_z), \tag{12}$$

we obtain equations for the potentials

$$\frac{d^2A}{dx^2} = \frac{4\pi e n_0}{c}\frac{eA}{Mc}\left[1 - \frac{2e\Phi}{Mv_{0i}^2} - \left(\frac{eA}{Mc}\right)^2\frac{1}{v_{0i}^2}\right]^{-1/2},$$

$$\frac{d^2\Phi}{dx^2} = -4\pi e n_0\left\{\left[1 - \frac{2e\Phi}{Mv_{0i}^2} - \left(\frac{eA}{Mc}\right)^2\frac{1}{v_{0i}^2}\right]^{-1/2} - e^{e\Phi/T}\right\}. \tag{13}$$

Here, n_0 is the particle number density at $x = -\infty$. $A \equiv A_y$, $A_z \equiv 0$, and T is the electron temperature in energy units.

Suppose that at the point $x = 0$ the quantity in the radical in Eq. (13) vanishes. This is then the turning point for the ions. Consequently, the system in (13) applies for $x < 0$; at $x > 0$,

$$d^2\Phi/dx^2 = 4\pi e n_0 e^{e\Phi/T}, \qquad d^2A/dx^2 = 0. \tag{14}$$

From Eqs. (13) and (14) and the boundary conditions we see that the qualitative behavior of the potentials A and Φ is that shown in Fig. 2.

In the region $x > 0$ Eq. (14) can be integrated with the boundary conditions given above, and we have

$$A = A_0 + H_0 x, \qquad x = D_e\,(e^{-e\Phi/2T} - e^{-e\Phi_0/2T}). \tag{15}$$

Here, A_0 and Φ_0 are the values of the potentials at $x = 0$ while $D_e = \sqrt{T/4\pi e^2 n}$ is the electron Debye radius. Thus, the effective thickness of electron sheath protruding into the region $x > 0$

*$\mathbf{v}\mathbf{A} = \mathbf{v}\cdot\mathbf{A}$.

is of the order of the Debye radius, as is to be expected.

In the region $x < 0$ the fields are described by (13); it is convenient to write these equations in the dimensionless form

$$\frac{d^2a}{d\xi^2} = \frac{a}{\sqrt{1-\alpha\psi-a^2}},$$

$$\frac{d^2\psi}{d\xi^2} = -\frac{Mc^2}{T}\left(\frac{1}{\sqrt{1-\alpha\psi-a^2}} - e^{\psi}\right) \qquad (16)$$

Here,

$$\psi = e\Phi/T, \quad a = eA/Mcv_{0i}, \quad \xi = x/D_{ic},$$
$$\alpha = 2T/Mv_{0i}^2; \quad D_{ic} = c/\omega_{0i}, \quad \omega_{0i}^2 = 4\pi e^2 n_0/M.$$

If $Mc^2/T \gg 1$, that is, if the electron energy is much less than the ion energy, the plasma can be considered neutral everywhere except in the region of the ion turning point, whose width must be of the order of the electron Debye radius. From the neutrality condition we obtain an equation which relates ψ and A:

$$e^{\psi} = 1/\sqrt{1-\alpha\psi-a^2}. \qquad (17)$$

If $\alpha \ll 1$, as we have assumed at the beginning, the equations for the potentials assume the form

$$d^2a/d\xi^2 = a/\sqrt{1-a^2}, \qquad (18a)$$

$$\psi = -\tfrac{1}{2}\ln(1-a^2). \qquad (18b)$$

Equation (18a) has the integral

$$\tfrac{1}{2}(da/d\xi)^2 = \text{const} - \sqrt{1-a^2}. \qquad (19)$$

For the boundary conditions $a|_{x\to-\infty} \to 0$, $\dot{a}|_{x\to-\infty} \to 0$ the constant of integration is found to be unity. Integrating Eq. (19) again with this condition, we have

$$\xi = \ln\left|\operatorname{tg}\frac{\chi}{4}\Big/\operatorname{tg}\frac{\pi}{8}\right| + 2\left(\cos\frac{\chi}{2} - \cos\frac{\pi}{4}\right), \qquad \sin\chi = a. \qquad (20)*$$

A similar relation has been obtained earlier by Ferraro[1] but in contrast with his work, in which a two-component plasma is considered, in our case the transition sheath is of the order of the ion Larmor radius rather than the electron Larmor radius.

B. We next consider the plasma system shown in Fig. 3, which might be called a single-Larmor sheath. As before, we assume that all ions have the same velocity and that this velocity is along the x axis at $x = 0$. We assume that the electrons, which compensate the charge, are cold.

This plasma system is described by (18) but the integral (19) now assumes the form

$$\dot{a}^2 = h_1^2 + 2(1-\sqrt{1-a^2}). \qquad (21)$$

Here,

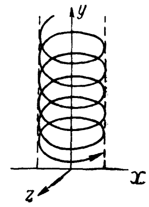

FIG. 3

$$h_1 = H_1 eD_{ic}/Mcv_{0i},$$

while H_1 is the strength of the magnetic field at $x = 0$.

Integrating Eq. (21) we obtain the variation of the field:

$$\xi = -4k^{-2}[E(k) - E(k, \pi/2-\chi)]$$
$$+2(1+2/k^2)[K(k) - F(k, \pi/2-\chi)], \qquad (22)$$

where $k^2 = 4/(4+h_1^2)$, $a = \sin\chi$ and $E(k)$, $E(k,\varphi)$, $K(k)$ and $F(k,\varphi)$ are elliptic integrals. The case $k = 1$ has been considered above. The case $k \to 0$, corresponding to $h_1 \to \infty$, can be easily computed by expanding (21) in powers of h_1^{-1}:

$$h = h_1 + h_1^{-1}\left(1-\sqrt{1-h_1^2\xi^2}\right). \qquad (23)$$

C. We now consider the boundary between the plasma and the field in the case in which the ions at $x = -\infty$ are characterized by a Maxwellian distribution in the presence of cold electrons that compensate the space charge (Fig. 1). The ion distribution function is assumed to be*

$$f = n_0(\pi c_T^2)^{-3/2}\exp\{-v^2/c_T^2\}. \qquad (24)$$

Here, n_0 is the particle number density at $x \to -\infty$ while c_T is the thermal velocity. Substituting Eq. (24) in Eq. (5) and introducing the dimensionless variables

$$a = eA/Mcc_T, \quad \eta_x = v_x/c_T,$$
$$\eta_y = v_y/c_T, \quad \xi^2 = x^2 4e^2 n_0/Mc_T^2,$$

after integration over v_z we obtain

$$\frac{d^2a}{d\xi^2} = \iint_G \eta_y \exp\{-(\eta_x^2+\eta_y^2)\}\, d\eta_x d\eta_y. \qquad (25)$$

The region of integration G is determined by (9) which, in our case, assumes the dimensionless form (cf. Fig. 4)

$$\eta_x^2 \geqslant a^2 + 2\eta_y a. \qquad (26)$$

Integrating the right-hand side of Eq. (25) over η_y we have

$$\ddot{a} = \frac{a}{2}\int_0^{\infty}\exp\{-a^2(1+\xi^2)^2/4\}\, d\xi. \qquad (27)$$

*tg = tan.

*In assuming that the electrons are cold, as a first approximation we assume that the electric field vanishes completely.

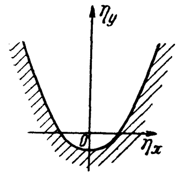

FIG. 4

Whence, using the boundary condition $a = 0$, $\dot{a} = 0$ at $x \to -\infty$, we have

$$\dot{a}^2 = \frac{\pi}{2} - 2\int_0^\infty \frac{\exp\{-a^2(1+\xi^2)^2/4\}}{(1+\xi^2)^2}d\xi. \quad (28)$$

Integrating once again, we find the explicit functional dependence $a = a(\xi)$:

$$a(\xi) = \sqrt{\frac{2}{\pi}}\int_0^a dx\left[1 - \frac{4}{\pi}\int_0^\infty \exp\{-x^2(1+\xi^2)^2/4\} \times d\xi/(1+\xi^2)^2\right]^{-1/2}. \quad (29)$$

The expression in the right-hand side of Eq. (29) is a universal function for a. In choosing the constant of integration in Eq. (29) we take account of the fact that at small values of a the field increases with coordinate according to a power relation and the point at which the field appears is taken to be the origin. Calculating Eq. (29) for small values of ξ, we have

$$a \approx 0.0028\,\xi^4, \quad (30)$$

whereas for large values of ξ

$$a \approx \xi\sqrt{\pi/2}. \quad (31)$$

A curve of Eq. (29) is shown in Fig. 5.

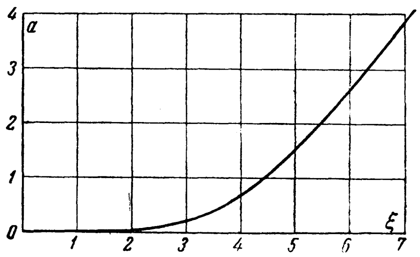

FIG. 5

3. MOTION IN AN AXIALLY SYMMETRIC FIELD

A. Suppose that a plasma cylinder is formed by monochromatic ions ($v = v_0$) and cold electrons characterized by a Maxwellian distribution ($T_e \ll Mv_i^2/2$). Suppose further that $H_\varphi = H_r = 0$, $H_z \neq 0$ and that all quantities depend only on r. If, in addition, we assume that $v_{iz} = 0$, the ion trajectories in this system will be of the form shown in Fig. 6a or Fig. 6b, depending on whether

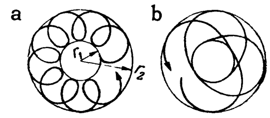

FIG. 6

the Larmor radius is greater than or less than the radius of the cylinder.

For the assumptions made above Eq. (11) becomes

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}rA_\varphi\right) = -\frac{4\pi Q}{cr}\left(\frac{P}{Mr} - \frac{eA_\varphi}{Mc}\right)\left[v_0^2 - \left(\frac{P}{Mr} - \frac{eA_\varphi}{Mc}\right)^2\right]^{-1/2} \quad (32)$$

Here, Q is the radial particle flux while P is the generalized momentum:

$$Q = enrv_r = \text{const}, \qquad P = r\,(Mv_\varphi + eA_\varphi/c) = \text{const}. \quad (33)$$

Introducing the dimensionless quantities

$$a = eA_\varphi/Mcv_0, \qquad p = P/Mv_0, \quad (34)$$

we obtain the following equation for a

$$\frac{d}{dr}\left(\frac{1}{r}\frac{dra}{dr}\right) = -\frac{4\pi eQ}{Mc^2v_0}\frac{p/r - a}{r\sqrt{1-(p/r-a)^2}}. \quad (35)$$

The magnetic field inside the cylindrical tube with inner radius r_1 and outer radius r_2 is determined by Eq. (35), where r_1 and r_2 are roots of the expression in the radical in Eq. (35). The field outside r_2 and inside r_1 is uniform. For a known value of a, the constant Q can be expressed in terms of the number of particles over the cross section of the cylinder N:

$$N = \int_{r_1}^{r_2} n2\pi r dr.$$

Using Eq. (33) we have

$$N = \frac{2\pi Q}{cv_0}\int_{r_1}^{r_2}\frac{dr}{\sqrt{1-(p/r-a)^2}}. \quad (36)$$

If the ion current is small, Eq. (35) can be solved by successive approximations. In the zeroth approximation the field is uniform and $A_\varphi = H_0 r/2$. In this case the quantity Q is given approximately by

$$Q = e^2H_0N/2\pi^2Mc. \quad (37)$$

Substituting the zeroth approximation for A_φ in the right-hand side of Eq. (35) and integrating, taking account of Eq. (37) we obtain an expression for the magnetic field H:

$$H = H_0\left\{1 - \frac{2e^2N}{\pi Mc^2}\arccos\frac{r_1r_2/r + r}{r_1 + r_2}\right\}, \quad r_1 \leqslant r \leqslant r_2. \quad (38)$$

For the case shown in Fig. 6a we must take $r_1 > 0$ in Eq. (38) whereas for the case shown in

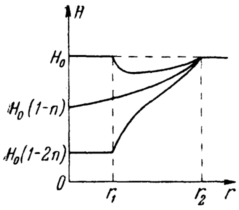

FIG. 7

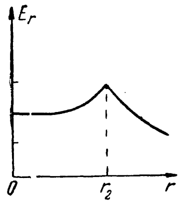

FIG. 8

Fig. 6b, in place of r_1 we must substitute $-r_1$. When $r_{1,2} \rightarrow \infty$ Eq. (38) becomes Eq. (23). Curves of the function in (38) for $r_1 > 0$, $r_1 < 0$ and $r_1 = 0$ are shown in Fig. 7. Attention is merited by the peculiar variation of the function $H(r, r_1, r_2)$ as r_1 goes through zero.

A similar result has been obtained by Tonks[5] by numerical integration of Eq. (35). This author has considered both dilute and dense plasmas. Later, Tonks[6] generalized these results to the case in which there is a spread of the momentum P.

Equation (38) shows that the small parameter used above in the method of successive approximations is the so-called "linear proton" $\Pi = e^2N/Mc^2$. It is not difficult to obtain terms in the expansion proportional to Π^2 and higher in Eq. (38).

We now consider the motion of particles in the cylindrical column. The equation of motion is

$$d\varphi/dr = v_\varphi / rv_r.$$

On the other hand, the field equation $(dH/dr = 4\pi j_\varphi/c)$ can be put in the following form [using (33)]:

$$dH/dr = -(4\pi Q/c)(v_\varphi / rv_r).$$

It follows that

$$H = -(4\pi Q/c)\,\varphi + \text{const.} \quad (39)$$

Taking account of Eq. (39) and the curves, we see that when $r_1 > 0$ (Fig. 6a), in the first approximation particles do not go around the axis of the system, whereas when $r_1 < 0$ (Fig. 6b) they do.

B. If the plasma is very dilute and the magnetic field is very strong, the preceding calculations can be applied to a cloud of fast particles in the absence of compensating electrons. In accordance with Eq. (33), the density distribution is related to the vector potential by the expression

$$n = \frac{Q}{ev_0}\frac{1}{r}\left[1-\left(\frac{p}{r}-a\right)^2\right]^{-1/2} \quad (40)$$

Assuming as the zeroth approximation that the magnetic field is uniform, $a = \alpha r$, we write the equation div $\mathbf{E} = 4\pi en$ in the form

$$\frac{1}{r}\frac{d}{dr}rE_r = \frac{4\pi Q}{rv_0}\left[1-\left(\frac{p}{r}-a\right)^2\right]^{-1/2}. \quad (41)$$

Integrating, and substituting the values of Q and p and α in terms of r_1 and r_2, we obtain the final formula for the electric field distribution inside the annular region:

$$E_r = \frac{2eN}{\pi r}\arccos\frac{r_1^2+r_2^2-2r^2}{r_2^2-r_1^2} \quad (42)$$

The field inside the ring $r < r_1$ vanishes while the field outside the cylinder is $E_r = 2eN/r$. In the case $r_1 = 0$ the electric field distribution is given by the expression

$$E_r = \frac{4eN}{\pi r_2}\frac{\arcsin(r/r_2)}{r/r_2} \quad (43)$$

and is shown in Fig. 8.

4. TWO-COMPONENT SYSTEMS

As an example of a two-component system we consider normal incidence of monochromatic fluxes of ions and electrons on a magnetic field. In general, the ion and electron velocities will be different at infinity, being designated by v_{0i} and v_{0e} respectively.* After substitution of the distribution functions

$$f_i = 2v_{0i}n_0\delta(v^2 + 2e\Phi/M - v_{0i}^2)\,\delta(v_y + eA/Mc)\,\delta(v_z),$$
$$f_e = 2v_{0e}n_0\delta(v^2 - 2e\Phi/m - v_{0e}^2)\,\delta(v_y - eA/mc)\,\delta(v_z) \quad (44)$$

Equation (5) assumes the following form in the region of common motion:

$$d^2\psi/d\xi^2 = (c^2/v_{0e}^2)\left[(1+\psi-a^2)^{-1/2} - (1-\theta\psi-\mu\theta a^2)^{-1/2}\right],$$
$$d^2a/d\xi^2 = a\left[(1+\psi-a^2)^{-1/2} + \sqrt{\mu\theta}\,(1-\theta\psi-\mu\theta a^2)^{-1/2}\right]. \quad (45)$$

Here

$$eA/mv_{0e}c = a, \quad 2e\Phi/mv_{0e}^2 = \psi, \quad \xi^2 = x^2 4\pi e^2 n_0/mc^2,$$
$$\mu = m/M, \quad \theta = mv_{0e}^2/Mv_{0i}^2.$$

The ions and electrons have turning points at

$$1+\psi-a^2 = 0, \quad 1-\theta\psi-\mu\theta a^2 = 0. \quad (46)$$

If $\mu\theta < 1$, the electrons come to rest first, while if $\mu\theta > 1$, the ions come to rest first.

If the plasma is nonrelativistic, then $c^2/v_{0e}^2 \gg 1$ and in solving (47) we may assume neutrality,† that

*The case $v_{0i} = v_{0e}$ has been considered by Ferraro.[1]

†In general, this procedure is valid only for pressures of approximately $H^2/8\pi$; it may not hold if the pressures are much smaller than $H^2/8\pi$.

is to say, $n_i = n_e$. Then the first equation in (45) gives a relation between ψ and a:

$$\psi = (1 - \mu\theta)\, a^2/(1 + \theta). \tag{47}$$

Substituting this value of ψ in the second equation of (45) we have

$$d^2 a_1 / d\xi_1^2 = a_1 / \sqrt{1 - a_1^2}, \tag{48}$$

where

$$a_1 = a\, [(1 + \mu)\theta/(1 + \theta)]^{1/2}, \qquad \xi_1 = \xi\, [1 + \sqrt{\mu\theta}]^{1/2}. \tag{49}$$

Equation (48) is identical with Eq. (18). In particular, this leads to the result that the thickness of the transition sheath is

$$\delta \sim (mc / 4\pi e^2 n_0)\, [1 + \sqrt{\mu\theta}]^{1/2}.$$

The quantity δ is of the order of electron Larmor radius in the field H_0 computed for the mean energy of the ions and electrons.

Similar conclusions can be obtained for a sheath of the type considered in Sec. 2C. Here, when neutrality obtains the thickness of the sheath is of the order of the electron Larmor radius. With a Maxwellian distribution the boundary between the plasma and the field, as before, will be of the order of the electron Larmor radius, but the actual pattern of the transition sheath cannot be expressed in simple analytic form.

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