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ON THE USE OF AN ARBITRARY GAUGE OF THE ELECTROMAGNETIC POTENTIALS IN THE DISPERSION METHOD

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The problem of the use of an arbitrary gauge of the electromagnetic potential of quantum electrodynamics within the framework of the dispersion method is considered. Several formulae are obtained which are generalizations of known expressions.

It is well known that owing to gauge invariance of quantum electrodynamics all physically meaningful quantities are independent of the gauge of the electromagnetic potentials. However there exist quantities in the theory which are not independent of the gauge, like, for example, the propagator, the vertex function, and diverse matrix elements. It is therefore desirable to discuss them in a general gauge. In particular, this facilitates the evaluation of these quantities.

In the dispersion method such an approach is difficult since there exists no unique method which would allow to determine the dependence on the gauge for a number of quantities. In particular, when writing matrix elements in covariant notation the summation over intermediate states, e.g. in reduction formulae, are usually carried out in the diagonal gauge ($d_l = 1$).

In the present paper* we shall study the systematic treatment of an arbitrary gauge in the dispersion method.

1. It is well known that the quantized vector potential of the free electromagnetic field† whose Lagrangian is given by

$$L(x) = -\frac{1}{2}\frac{\partial A_\mu(x)}{\partial x^\nu}\frac{\partial A_\mu(x)}{\partial x^\nu} \qquad (1.1)$$

can be written in the form

$$A_\mu(x) = (2\pi)^{-3/2}\int d^4k\sqrt{2k_0}\,\theta(k_0)\,\delta(k^2)\,e_\mu^\lambda\,\{e^{ikx}c_\lambda^+(\mathbf{k}) + e^{-ikx}c_\lambda^-(\mathbf{k})\}, \qquad (1.2)$$

while the commutation relations for the operators $c_\lambda^\pm(\mathbf{k})$ follow from the form of the Lagrangian (the structure of the operator is that of the energy-momentum 4-vector P_μ), from the requirement of a positive energy and from translational invariance:

$$\partial A_\mu(x)/\partial x^\nu = i\,[P_\nu, A_\mu(x)]. \qquad (1.3)$$

The operator $A_\mu(x)$ is determined up to a gauge transformation

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial f(x)/\partial x^\mu, \qquad (1.4)$$

where $f(x)$ is, generally speaking, an arbitrary operator function. Expressing $A_\mu(x)$ in terms of $A'_\mu(x)$ by means of the implicit equation (1.4) and inserting in (1.1) we obtain

$$L(A_\mu(x)) = L'(A'_\mu(x)), \qquad (1.5)$$

where L' indicates the change in the form of the Lagrangian induced by the gauge transformation (1.4). As a result of the change of $L(x)$ the equation for $A_\mu(x)$ changes into

$$\Box A'_\mu(x) = -\partial\Box f(x)/\partial x^\mu. \qquad (1.6)$$

We note that in a gauge transformation both the change of $A_\mu(x)$ to $A'_\mu(x)$ and the change indicated by (1.5) has to be considered in order to maintain the original quantization scheme, i.e., the meaning and the commutation relations of the operators $c_\lambda^\pm(\mathbf{k})$. In particular, then also the form of the energy-momentum vector P_μ remains unchanged.

We now are going to find a class of functions $f(x)$ which obey the following conditions: (i) the operators $A'_\mu(x)$ are Hermitian; (ii) the operators $A'_\mu(x)$ are translationally invariant:

$$\partial A'_\mu(x)/\partial x^\nu = i\,[P_\nu, A'_\mu(x)]; \qquad (1.7)$$

(iii) the gauge transformation (1.4) has to lead to the known expression for the photon propagator

*The basic results of this work have already been utilized in a previous paper[1] by the present authors.

†Due to the gauge invariance of the S-matrix (see reference 2, p. 247) the results obtained below are valid also for interacting fields.

$$D_{\mu\nu}(k) = -\frac{1}{k^2 + i\varepsilon}\left\{\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + d_l \frac{k_\mu k_\nu}{k^2}\right\}, \quad (1.8)$$

where the range of the quantities d_l has to include at least the values $d_l = 0$ and $d_l = 1$. These correspond to the important cases of the transversal and diagonal gauge respectively.

These requirements substantially restrict the form of $f(x)$. As can be easily seen, (1.7) leads to a linear dependence of $f(x)$ on the field operators. Condition (i) restricts $f(x)$ to the class of functions of the form

$$\partial f(x)/\partial x^\mu = (2\pi)^{-3/2}\int d^4k \sqrt{2k_0}\, \theta(k_0)\, \delta(k^2) \times k_\mu (e^\lambda k)\, f_\pm(k^2)\, [e^{ikx} c^+_\lambda(\mathbf{k}) \pm e^{-ikx} c^-_\lambda(\mathbf{k})], \quad (1.9)$$

where $f_+(k^2)$ and $if_-(k^2)$ are arbitrary real functions. Condition (iii) allows to connect $f_\pm(k^2)$ with the longitudinal part of $D_{\mu\nu}(k)$. We obtain immediately (see reference 2, p. 259)

$$f_+(k^2) = (\sqrt{d_l} - 1)/k^2, \qquad if_-(k^2) = \sqrt{d_l - 1}/k^2, \quad (1.10)$$

and the requirement $d_l \geq 0$ eliminates the second possibility.

From (1.9) and (1.10) one sees that $\partial \Box f(x)/\partial x^\mu \neq 0$, and thus (1.6) differs considerably from the equation for $A_\mu(x)$. We do not consider here the specialized gauge transformation with $\Box f(x) = 0$.

The relations (1.2), (1.4), (1.9), and (1.10) allow to write $A'_\mu(x)$ in the form

$$A'_\mu(x) = (2\pi)^{-3/2}\int d^4k \sqrt{2k_0}\, \theta(k_0)\, \delta(k^2)\, \tilde{e}^\lambda_\mu(k)\, \{e^{ikx} c^+_\lambda(\mathbf{k}) + e^{-ikx} c^-_\lambda(\mathbf{k})\}, \quad (1.11)$$

where

$$\tilde{e}^\lambda_\mu(k) = e^\lambda_\mu + k_\mu k^{-2} (e^\lambda k)(\sqrt{d_l} - 1). \quad (1.11')$$

The pole $k^2 = 0$ in the integrals of the form

$$\int d^4k \varphi(k)\, \delta(k^2)/k^2$$

is understood to be taken as a principal value. In actual calculations it is convenient to utilize

$$\mathrm{P}x^{-1}\delta(x) = -\delta'(x).$$

We note that due to

$$\partial A'^{\pm}_\mu(x)/\partial x^\mu = \sqrt{d_l}\, \partial A^\pm_\mu(x)/\partial x^\mu \quad (1.12)$$

the admissible states remain unchanged, and the relation

$$\langle \Phi_{\text{adm}}, A'_\mu(x)\, \Phi_{\text{adm}}\rangle = \langle \Phi_{\text{adm}}, A_\mu(x)\, \Phi_{\text{adm}}\rangle \quad (1.13)$$

insures the correspondence with Maxwell's equations. Indeed, one sees from (1.13) that the appearance of d_l in the propagator $D_{\mu\nu}(k)$ is associated with a change in the unphysical part of $A_\mu(x)$.

We can thus say that a gauge transformation which obeys the above specified conditions turns out to change the polarization vectors e^λ_μ into $\tilde{e}^\lambda_\mu$ in the expansion (1.2) of the operators $A_\mu(x)$ in terms of the operators $c^\pm_\lambda(\mathbf{k})$. In the following we shall omit the prime in $A'_\mu(x)$.

It is easy to see that the dependence of matrix elements on d_l when separating them out from the dependence on photons is exactly the same. For example, the matrix element $\langle \alpha | F | \beta;\, \mathbf{k}\lambda\rangle$ of an arbitrary operator F (the indices $\mathbf{k}$ and λ denote a photon of momentum $\mathbf{k}$ and polarization λ) in the diagonal gauge due to considerations of covariance can be written in the form $\langle \alpha | F | \beta;\, \mathbf{k}\lambda\rangle = e^\lambda_\mu F_\mu$. After a gauge transformation this changes to $\langle \alpha | F | \beta;\, \mathbf{k}\lambda\rangle = \tilde{e}^\lambda_\mu(k)\, F_\mu(d_l)$. In case of gauge invariance of F_μ the whole dependence of the matrix element $\langle \alpha | F | \beta;\, \mathbf{k}\lambda\rangle$ on d_l is contained in $\tilde{e}^\lambda_\mu(k)$.

2. When investigating matrix elements it is frequently necessary to bring them into a covariant form. The use of an arbitrary gauge introduces certain changes into the expressions which give the operators $c^\pm_\lambda(\mathbf{k})$ in terms of $A_\mu(x)$. One can easily show that

$$\tilde{e}^\lambda_\mu(\mathbf{k})\, c^\pm_\lambda(\mathbf{k}) = \frac{\mp i}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k_0}} \int d\mathbf{x} e^{\mp ikx} \frac{\overleftrightarrow{\partial}}{\partial x^0} A_\mu(x). \quad (2.1)$$

In order to split off $c^\pm_\lambda(\mathbf{k})$ we introduce the vectors $\bar{e}^\lambda_\mu(k)$ which are taken to be orthogonal to $\tilde{e}^\lambda_\mu(k)$:

$$\bar{e}^\lambda_\mu(k) = e^\lambda_\mu + k_\mu (e^\lambda k)\, k^{-2}(1/\sqrt{d_l} - 1). \quad (2.2)$$

Indeed,

$$\bar{e}^\lambda_\mu(k)\, \tilde{e}^\lambda_\nu(k) = \delta_{\mu\nu}; \qquad \bar{e}^\lambda_\mu(k)\, \tilde{e}^\nu_\mu(k) = \delta_{\lambda\nu}. \quad (2.3)$$

From (2.1) and (2.3) we obtain

$$c^\pm_\lambda(\mathbf{k}) = \frac{\mp i}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k_0}}\, \bar{e}^\lambda_\mu(k) \int d\mathbf{x} e^{\mp ikx} \frac{\overleftrightarrow{\partial}}{\partial x^0} A_\mu(x). \quad (2.4)$$

This way one has to exchange the polarization vectors e^λ_μ for $\bar{e}^\lambda_\mu(k)$ in the usual reduction formulae when one goes to arbitrary gauge.

However, it is more convenient to use (2.4) in a somewhat changed form, namely, in terms of $\tilde{e}^\lambda_\mu$ instead of $\bar{e}^\lambda_\mu$ (see the concluding remarks of section 1.). Using the relation

$$\bar{e}^\lambda_\mu(k) = \tilde{e}^\lambda_\mu(k) + k_\mu (e^\lambda k)\, k^{-2}(1/\sqrt{d_l} - \sqrt{d_l}), \quad (2.5)$$

we obtain from (2.4)

$$c^\pm_\lambda(\mathbf{k}) = \mp \frac{i}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k_0}}\, \tilde{e}^\lambda_\mu(k) \int d\mathbf{x} e^{\mp ikx} \frac{\overleftrightarrow{\partial}}{\partial x^0} A_\mu(x) \mp \frac{i}{(2\pi)^{3/2}} \frac{k_\mu}{\sqrt{2k_0}} \frac{(e^\lambda k)}{k^2}\left(\frac{1}{\sqrt{d_l}} - \sqrt{d_l}\right) \int d\mathbf{x} e^{\mp ikx} \frac{\overleftrightarrow{\partial}}{\partial x^0} A_\mu(x). \quad (2.6)$$

When utilizing (2.4) and (2.6) to bring matrix elements into covariant form there appear different Green's functions which can be expressed in terms of G, $D_{\mu\nu}$ and Γ_μ. These, and also the second term of (2.6) can be brought to a simpler form by means of the relations[3]

$$k_\mu D_{\mu\nu}(k) = -k_\nu d_l/(k^2 + i\varepsilon) \quad ,$$
$$k_\mu \Gamma_\mu(p, p-k) = G^{-1}(p) - G^{-1}(p-k). \qquad (2.7)$$

3. In the dispersion method, frequent use is made of expansions into intermediate states, in which the eigenfunctions of the free field energy-momentum operator are taken as the complete set of states. In quantum electrodynamics such a set of intermediate states is taken which besides the transverse photons contains also the longitudinal and scalar photons. Therefore in the sum over intermediate states

$$\langle\alpha|AB|\beta\rangle = \Sigma\langle\alpha|A|n\rangle\langle n|B|\beta\rangle$$

the summation over the polarization λ goes from 0 to 3. Writing the matrix elements of A and B which contain the polarization index in arbitrary gauge in the form

$$\tilde{e}^\lambda_\mu(k)A_\mu, \qquad \tilde{e}^\lambda_\mu(k)B_\mu,$$

we find that instead of $e^\lambda_\mu e^\lambda_\nu = \delta_{\mu\nu}$ there appears, according to (1.11′),

$$\tilde{e}^\lambda_\mu(k)\,\tilde{e}^\lambda_\nu(k) = (\delta_{\mu\nu} - k_\mu k_\nu/k^2) + d_l k_\mu k_\nu/k^2.$$

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[2] N. N. Bogolyubov and D. V. Shirokov, Введение в теорию квантованных полей (Introduction into the Theory of Quantized Fields), interscience, 1959.

[3] E. S. Fradkin, JETP **29**, 258 (1955), Soviet Phys. JETP **2**, 361 (1956).

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