STRUCTURE OF THE TRANSITION LAYER BETWEEN A PLASMA AND A MAGNETIC FIELD

V. P. SHABANSKII

Institute for Nuclear Physics, Moscow State University

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The structure of the region in which a plasma beam is reflected by a magnetic field has been investigated using the self-consistent microscopic equations of motion for the particles; the structure of the transition layer between a fixed plasma and a magnetic field has also been investigated. In the first case corrections are introduced to take account of the polarization which is produced at high velocities of the incident beam.

1. In a number of problems concerned with the motion of plasma in magnetic fields, it is found that the plasma region and the region occupied by the magnetic field remain separated by boundary regions which exist for finite lengths of time. It is of practical interest to determine the thickness of these transition layers and the variation of particle density and magnetic field inside the layers. It is well-known that the diamagnetism of a plasma with infinite conductivity, which shields it from an external field, is due to true surface currents. If the layer is thin and if the velocity of the beam incident on the magnetic wall is high, the electrons that comprise the shielding current can be relativistic.

The analysis of retardation of a plasma bunch of finite dimensions in a magnetic field has much in common with a problem treated by Veksler.[1] The fact that the kinetic energy of the bunch is converted at the turning point primarily into transverse motion of the electrons[1]

$$w_{\perp}^{-} = m^{-} (v_{\perp}^{-})^2/2 = m^{+} (v_0^{+})^2/2 \qquad (1)$$

(m^- and m^+ are the mass of the electron and ion respectively, $v_{\perp}^-$ is the electron velocity perpendicular to the motion of the bunch, v_0 is the bunch velocity before retardation) has been noted by Chapman and Ferraro (cf., for example, reference 2).

An extension of Eq. (1) into the relativistic region by simply taking account of the increase in electron mass m^-, as in reference 1, is inexact. Actually, at high velocities of the incident beam (v_0) it is necessary to take account of the polarization of the plasma in the direction of incidence; this process consumes part of the kinetic energy which, in turn, does not enter into the transverse energy of the electrons. Polarization also leads to another effect: the distribution of the longitudinal kinetic energy which is converted into transverse energy of ions and electrons is affected and this effect also acts to reduce the value of $w_{\perp}^-$ as compared with that given in Eq. (1). For this reason, at relativistic velocities the transverse energy of the electrons will be smaller than the value which follows from Veksler's analysis, which assumes a relativistic electron mass.[1]

2. We first consider the motion of two particles of different mass in the xy plane in a uniform magnetic field H_0 along the z axis; we assume rigid coupling in the x direction and no coupling in the y direction. The self-field is neglected. This model actually corresponds to the motion of particles in a plasma which is incident from $x = -\infty$ and which is reflected from a magnetic wall in the region $x \geq 0$. The plasma is assumed to be dense enough so that the electric polarization forces in the x direction are large, but rarefied enough so that the field H_0 is not distorted. The equations of motion for each of the particles

$$m\frac{dv_x}{dt} = e\left(E_x + \frac{1}{c}v_y H_0\right), \qquad m\frac{dv_y}{dx} = -\frac{e}{c}v_x H_0 \qquad (2)$$

(the conventional notation is used), with rigid coupling in the x direction $v_x^+ = v_x^-$ and the initial condition $v_x^- = v_x^+ = v_0$ at $t = 0$, show that the particle-velocity vectors rotate along appropriate ellipses at the same frequency:

$$v_x = v_x^- = v_x^+ = v_0 \cos\omega t,$$
$$v_y^- = v_{y0}^- - \sqrt{m^+/m^-}\, v_0 \cos(\omega t + \pi/2),$$
$$v_y^+ = v_{y0}^+ + \sqrt{m^-/m^+}\, v_0 \cos(\omega t + \pi/2),$$
$$\omega = eH_0/c\sqrt{m^+m^-} = \sqrt{\omega^-\omega^+}. \qquad (3)$$

Here, ω^+ and ω^- are the Larmor frequencies for the ions and electrons respectively in the field H_0 while $v_0^\pm$ is the initial velocity. Integration of Eq. (3) over a quarter cycle $[t_0 = \pi/2\omega]$ for $v_{y0}^+ = v_{y0}^- = 0$ yields the appropriate semi-axes of the ellipses described by the particles

$$r_0 = x(t_0) = \int_0^{t_0} v_x dt = \sqrt{r^- r^+} = \sqrt{m^- m^+}\, c v_0 / eH_0, \quad y^+(t_0) = -r^-, \quad y^-(t_0) = r^+. \tag{4}$$

Thus, the depth of penetration of the particles into the region of magnetic field is equal to the mean geometric value of the Larmor radii r^+ and r^- while the path traversed by the electron along the boundary $y^-(t_0)$ is m^+/m^- times greater than the path traversed by the ion (cf. Fig. 1). It is apparent from Eq. (3) that at $t_0 = \pi/2\omega$ the proton and the electron exchange kinetic energies:

$$w_\perp^+(t_0) = m^- v_0^2 / 2, \qquad w_\perp^-(t_0) = m^+ v_0^2/2.$$

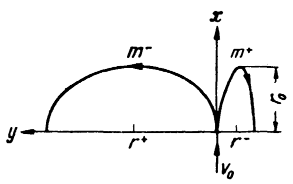

FIG. 1. Trajectories of particles m^+ and m^- in a magnetic field; the field is directed toward the reader and occupies the region $x \geq 0$.

This problem is equivalent to the problem of reflection of plasma from a magnetic wall H_0 when the conditions noted above are satisfied (rarefied plasma and absence of polarization); these conditions are stipulated by

$$m^+ N_0 v_0^2 \ll H_0^2 / 8\pi, \qquad \delta = \sqrt{m^+ v_0^2 / 4\pi N_0 e^2} \ll r_0,$$

where δ is the Debye polarization length for a particle of energy $m^+ v_0^2$ while r_0 is determined by Eq. (4). These requirements imply

$$m^+ N_0 v_0^2 \ll H_0^2 / 8\pi \ll m^- c^2 N_0. \tag{5}$$

If the left-hand equality is not satisfied it is necessary to take account of the self magnetic field of the current in the y direction; if the right-hand relation is not satisfied polarization must be taken into account. In particular, it can be seen that if the inverse inequality holds, i.e., $H_0^2/8\pi \gg m^- c^2 N_0$, the coupling between the particles in the x direction is so small that each particle moves in a circle with the appropriate Larmor radius and, in general, there is no transfer of energy.

We show below that if the field is free (that is to say, if in the absence of the plasma it extends over all space) the condition for stationary reflection for a plasma of infinite extent leads to the condition $m^+ v_0^2 N_0 = H_0^2/8\pi$, while the inequality in (5), which allows us to neglect polarization, reduces to the requirements $m^+ v_0^2 / m^- c^2 \ll 1$ or $w_\perp^-(t_0)/m^- c^2 \ll 1$ i.e., the electron velocities $v_\perp$ must be nonrelativistic.

3. A sufficiently small portion of the surface which divides the regions occupied by the magnetic field and the plasma may be regarded as a plane surface; this is taken to be the plane $x = \text{const}$. The problem is formulated as follows. A plasma beam is incident from the region $x = \infty$ in the $-x$ direction, with a given velocity. At $x = -\infty$ there is a field H_0 which is along z; the magnitude of this field is to be determined from the existence condition for the stationary solution.

It is clear from the symmetry of the problem that all quantities can only vary in the x direction and that there are only two particle-velocity components, $v_x^\pm = v^\pm$ and $v_y^\pm$. It will be assumed that the velocities of the ion beam (+) and the electron beam (−) can be different at infinity. This means that in addition to considering reflected plasma beams we can, in rough approximation treat also a fixed plasma. In the first case, the velocities of the electrons and ions are equal at infinity and can be identified with a macroscopic beam velocity which is large enough so that the thermal velocities can be neglected. (A similar formulation of the problem has been considered by Chapman and Ferraro.[2]) In the second case (fixed plasma) the velocities of the particles can be identified with the mean thermal velocities, whose dispersion in direction and absolute magnitude is negligible (i.e., the fixed plasma is approximated by two interpenetrating beams.

The complete system of equations of motion for the particles in the self-consistent fields is

$$m^+ v^+ \frac{dv^+}{dx} = e\left(E + \frac{1}{c} v_y^+ H\right),$$

$$m^- v^- \frac{dv^-}{dx} = -e\left(E + \frac{1}{c} v_y^- H\right),$$

$$m^+ dv_y^+/dx = -eH/c, \qquad m^- dv_y^-/dx = eH/c;$$

$$j_y = eN^+ v_y^+ - eN^- v_y^- = -\frac{c}{4\pi}\frac{dH}{dx}, \quad \frac{dE}{dx} = 4\pi e (N^+ - N^-),$$

$$N^+ v^+ = N_0 v_0^+, \qquad N^- v^- = N_0 v_0^- . \tag{6}$$

Here, N_0 is the plasma density at infinity (for both beams, incident and reflected, so that the density in one beam is $N_1 = N_0/2$); v_0^+ and v_0^- are the velocities for the positive and negative particles at x

$= +\infty$. The positive direction for the velocity is taken to be from the plasma toward the boundary i.e., in the negative x direction; the remaining notation is conventional.

The temperature and pressure in the x and y directions in the second problem (fixed plasma) are given by the formulas

$$p^{\pm}_{x,\,y} = kT^{\pm}_{x,\,y}N^{\pm} = N^{\pm}m^{\pm}(v^{\pm}_{x,\,y})^2. \qquad (7)$$

We now convert to dimensionless variables in which v, N, H and x are measured in units of

$$v_0^+, \quad N_0, \quad H_1 = (4\pi N_0 m^+)^{1/2} v_0^+,$$
$$x_1 = (m^- c^2 / 4\pi e^2 N_0)^{1/2}. \qquad (8)$$

Then, eliminating E, N^+, N^-, v_y^+ and v_y^- in (6), we obtain the system*

$$\sqrt{\frac{\alpha}{\beta}}\, v^- \frac{dH}{dx} = \left(1 + \alpha \sqrt{\frac{\alpha}{\beta}\frac{v^-}{v^+}}\right) \int_\infty^x H dx,$$
$$v^+ \frac{dv^+}{dx} + \alpha v^- \frac{dv^-}{dx} = -(1+\alpha)\, H \int_\infty^x H dx,$$
$$\gamma_\alpha v^- \frac{dv^-}{dx} + \gamma H \int_\infty^x H dx = -\int_\infty^x \left(\frac{1}{v^+} - \frac{1}{\sqrt{\alpha/\beta}\, v^-}\right) dx, \qquad (9)$$

where

$$\alpha = m^- / m^+, \quad \beta = m^-(v_0^-)^2 / m^+ (v_0^+)^2 = T_-/T_+,$$
$$\gamma = m^+ (v_0^+)^2 / m^- c^2. \qquad (10)$$

The particle velocities in the y direction are:

$$v_y^- = \frac{1}{\sqrt{\alpha}} \int_\infty^x H dx, \qquad v_y^+ = -\sqrt{\alpha} \int_\infty^x H dx. \qquad (11)$$

In the incident-beam problem, we must put $\beta = \alpha$ in (9); in the case of a fixed plasma with equal electron and ion temperatures at infinity, we put $\beta = 1$. It is clear from Eq. (11) that the velocity ratio $v_y^-/v_y^+ = -m^+/m^-$ is the independent of γ and corresponds to the condition that the total y momentum must vanish provided $N^-(x) = N^+(x)$. However, in the presence of polarization the total momentum does not vanish and there is motion of matter in the y direction.

4. In the absence of polarization ($\gamma \ll 1$) the equations can be solved easily. From the third equation in (11) we find that $v^+ = \sqrt{\alpha/\beta}\, v^-$, $N = N^+ = N^- = 1/v^+$ and the system reduces to two equations for v^+ and H:

$$v^+ \frac{dH}{dx} = \int_\infty^x H dx, \qquad v^+ \frac{dv^+}{dx} = -H \int_\infty^x H dx, \qquad (12)$$

*If we use the vector and scalar field potentials and eliminate V^+ and V^- rather than E, the system is somewhat more compact; however, the form used in the text is more convenient for the introduction of relativistic corrections.

where H is measured in units of $H_1\sqrt{1+\beta}$ while x is measured in units of $x_1/\sqrt{1+\alpha}$.

The first integral of these equations gives a relation between v^+ and H:

$$v^+ = 1 - H^2/2 \qquad (13)$$

or, in terms of the usual dimension variables:

$$v^+ = v_0^+ - H^2 / 8\pi m^+ v_0^+ N_0 (1+\beta). \qquad (13')$$

For the incident-beam problem[2] we have $\beta = \alpha \ll 1$, $v_0^- = v_0^+ = v^0$ and Eq. (13′) yields the expression

$$v = v_0 - H^2 / 8\pi m^+ v_0 N_0.$$

The thickness of the transition layer is approximately $x_1/\sqrt{1+\alpha}$. For a fixed plasma Eq. (13′) together with the relation $v^- = \sqrt{\beta/\alpha}\, v^+$ determines the relations between the electron-ion pressure and the field. Using (7) for the pressure and the equation of continuity, we have from Eq. (13′)

$$p_x = p_x^0 - H^2/8\pi, \qquad p_x = p_x^- + p_x^+, \qquad p_x^-/p_x^+ = T_x^-/T_x^+ = \beta; \qquad (13'')$$

the first two of these equations coincide with the results of the hydrodynamic analysis.

The value of the magnetic field at the turning point ($v = 0$) at $x = 0$ is given by

$$H_0^2/8\pi = m^+ (v_0^+)^2 N_0 (1+\beta). \qquad (14)$$

If Eq. (14) is used to eliminate N_0 in Eq. (8) for x_1, which characterizes the thickness of the transition layer, we have

$$x_1 = [2 r_0^+ r_0^- \sqrt{\alpha/\beta}\,(1+\beta)]^{1/2},$$

where r_0^- and r_0^+ are the Larmor radii of rotation for electrons and ions with velocities v_0^- and v_0^+ in the field H_0. In the case of an incident beam ($\beta = \alpha \ll 1$) this expression coincides with (4) for r_0 to within a factor $\sqrt{2}$. The difference is explained by the fact that in the present case the particles move in an inhomogeneous self-consistent field.

Substituting Eq. (12) in Eq. (13), we obtain an equation for the magnetic field in the layer:*

*We arrive at the same equation in the case of a plasma of equal masses ($m^- = m^+$). We put in Eq. (9) $\alpha = \beta = 1$. Furthermore, in order for this plasma density to coincide with the density of a real electron-ion plasma, we must write in the equation of continuity $N_0/2$ in place of N_0. Then it is apparent that $v^- = v^+$ for any value of γ and the equations in (9) reduce to two identical systems, each of which reduces to Eq. (15). The difference lies only in the fact that we introduce in Eq. (8) for x_1 m^+ in place of m^-, that is to say, the magnitude of the field for which stationary reflection is possible remains the same [Eq. (14)] but the width of the transition layer is increased by a factor of $\sqrt{m^+/m^-}$. This is explained by the absence of the constraining effect of the electrons.

$$\frac{d}{dx}\left\{\left(1-\frac{H^2}{2}\right)\frac{dH}{dx}\right\} = H, \tag{15}$$

with the boundary conditions $H = 0$ at $x = \infty$ and $H = \sqrt{2}$ at $x = 0$.* Introducing the variable $Z = dx/dH$ we can write Eq. (15) in the form

$$\frac{d}{dH}\left\{\left(1-\frac{H^2}{2}\right)\frac{1}{Z}\right\} = ZH, \qquad Z = \frac{dx}{dH}, \tag{16}$$

with the boundary condition $Z = -\infty$ at $H = 0$. Integrating Eq. (16) and choosing from among the two solutions the one which satisfies the boundary condition, we have

$$Z = -(1 - H^2/2)/H\sqrt{1 - H^2/4}, \tag{17}$$

$$x = \frac{1}{2}\ln\frac{(\sqrt{2}-1)(1+\sqrt{1-H^2/4})}{(\sqrt{2}+1)(1-\sqrt{1-H^2/4})} - 2\sqrt{1-\frac{H^2}{4}} + \sqrt{2}. \tag{18}$$

The velocities $v^- = \sqrt{\beta/\alpha}\, v^+$, $v^+ = 1 - H^2/2$ in the x direction are determined from the equations

$$x = \frac{1}{2}\ln\frac{(\sqrt{2}-1)(\sqrt{2}+\sqrt{1+v^+})}{(\sqrt{2}+1)(\sqrt{2}-\sqrt{1+v^+})} - \sqrt{2(1+v^+)} + \sqrt{2}. \tag{19}$$

In Fig. 2 we show the approximate behavior of H, v and $vH = H(1 - H^2/2)$ for the case of a reflected beam ($\beta = \alpha \ll 1$) as a function of the layer depth x. The velocities and energies in the y direction can be found from Eq. (11).

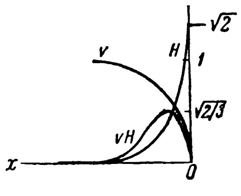

FIG. 2. Behavior of the quantities v, H and vH in the transition layer.

In particular, at the turning point $x = 0$, $H = \sqrt{2}$, $\int_\infty^x H\,dx = 1$, the velocities and energies in the ordinary variables (when $\alpha \ll 1$) are given respectively by

$$v_y^+ = \sqrt{(1+\beta)\,m^-/m^+}\, v_0^+, \qquad v_y^- = -\sqrt{(1+\beta)\,m^+/m^-}\, v_0^+,$$
$$w_y^+ = m^-(v_0^+)^2(1+\beta), \qquad w_y^- = m^+(v_0^+)^2(1+\beta). \tag{20}$$

When $\beta = \alpha \ll 1$ (incident beam) the expressions in (20) show that the electrons and ions exchange kinetic energies at the turning point.

At the turning point the velocities v^+ and v^- vanish so that the particle density N and the current density along the y axis become infinite.

*If we neglect the term $H^2/2$ compared with unity in Eq. (15) then this equation coincides with the equation that describes the penetration of a magnetic field into a superconductor. This equation is simpler for the superconductor because the density of the superconducting electrons is assumed to be constant over the entire depth of the layer.

This effect arises because we have neglected the directional dispersion in the velocity. The mean values of those quantities remain finite in the layer. In order to compute the mean values we must know the number of particles in the layer. This quantity can be determined as follows:*

$$n = \int_0^\infty N v_y^+\, dx \Bigg/ \int_0^\infty v_y^+\, dx = \int_{\sqrt{2}}^0 \frac{v_y^+}{v^+} Z\,dH \Bigg/ \int_{\sqrt{2}}^0 v_y^+ Z\,dH$$
$$= \frac{3}{2\sqrt{2}-1} \approx 1.66. \tag{21}$$

It is then easy to show that the mean velocities will be $n/\sqrt{2} \sim 1.2$ times smaller than the maximum values given by Eq. (20).

The total current in the y direction can be determined either from the equation for j in (6) or from the expression $I \approx I^- = -enN_0\overline{v_y}x$. We find†

$$I = cv_0^+\sqrt{(1+\beta)\,N_0 m^+/2\pi}. \tag{22}$$

5. We now consider the case in which polarization must be taken into account. In this case, only the incident-beam problem has any meaning, since the temperature of a fixed plasma at $\gamma = 1$ would be $T^+ \sim 10^9$ deg. On the other hand, beams with velocities $v_0 = \sqrt{m^- c^2/m^+} \sim 7 \times 10^8$ cm/sec, corresponding to $\gamma = 1$, are completely feasible. Hence we can write in Eq. (11) $\alpha = \beta \ll 1$, thereby obtaining the system of equations

$$v^-\frac{dH}{dx} = \int_\infty^x H\,dx, \quad v^-\frac{dv^+}{dx} = -H\int_\infty^x H\,dx, \quad \frac{1}{v^-}$$
$$-\frac{1}{v^+} = \gamma\frac{d}{dx}\left\{H\int_\infty^x H\,dx\right\}. \tag{23}$$

We shall only find the correction to the velocities obtained in the preceding section, assuming that $\gamma < 1$. In the first approximation ($\gamma = 0$) we have the case already treated, where $v = v^+ = v^-$, $N = N^+ = N^-$, and the magnetic field is given by Eq. (18). In the second approximation, the field remains the same but the mean values of the velocities $\overline{v^-}$ and $\overline{v^+}$ are determined from the third equation in (23).

Using the relations obtained in Sec. 4 we find:

$$\overline{v^-} = \overline{v^+}(1 - k_1\gamma), \tag{24}$$

k_1 is a numerical factor of order unity.

*If n is determined by the expression,

$$n = \int_0^\infty NH\,dx \Bigg/ \int_0^\infty H\,dx,$$

then we find $n = \pi/2 \sim 1.6$, which coincides with Eq. (21) in order of magnitude.

†The mean current density $\bar{j} = I/x_1 = cH/4\pi\sqrt{2}\,x_1$ coincides with the expression for the density of surface current in a superconductor in a magnetic field H.

Multiplying the third equation in (23) by v_y^- and integrating over the entire layer, we obtain expressions for the mean velocity and energy of electrons, taking account of polarization in the x direction:

$$\overline{(v_y^-)}_\gamma = \overline{v_y^-}(1 - k_2\gamma),$$
$$\overline{(w_y^-)}_\gamma = \overline{w_y^-}(1 - k_3\gamma) = \frac{3}{4n}\frac{m^+ v_0^2}{2}(1 - k_3\gamma), \quad (25)$$

where k_2 and k_3 are numerical factors approximately equal to unity.

Thus, the relation $w_y^- = m^+(v_0^+)^2/2$, which applies for low beam velocities, is not satisfied here. The kinetic energy of the electrons along the y axis is smaller than this value for $\gamma \neq 0$. In the second approximation the total energy of the electrons and ions along the y axis is equal to the sum of the energies of the electrons and ions in the beam. This is apparent from the relation $\sqrt{\alpha}\, v_y^- = v_y^+/\sqrt{\alpha}$, which is satisfied for any γ, and from the fact that the field H does not change in the second approximation, so that

$$\sqrt{\alpha}\, v_y^- = \int_\infty^0 H dx = -1.$$

Hence, the effect of polarization is not primarily to change the total kinetic energy of the particles, but rather to affect the redistribution of energy between electrons and ions in the layer in such a way that the transverse energy of the electrons is reduced while that of the ions is increased as compared with the values in the absence of polarization. The effect of polarization on the total energy along the y axis first appears in the next approximation in γ. In this case the thickness of the layer increases and the magnetic field is changed so that in absolute magnitude

$$\sqrt{\alpha}\, v_y^- = v_y^+ / \sqrt{\alpha} = \left| \int_\infty^0 H dx \right| < 1.$$

The remainder of the kinetic energy of the particle goes into polarization energy.

Both effects appear before the relativistic increase in mass; hence in the analysis of retardation of a plasma bunch in an axially symmetric magnetic field[1] with $\gamma \gtrsim 1$ these effects should be considered on equal terms with the latter. The account of these effects should lead to a reduction in the acceleration of electrons in the collision of a fast plasma beam with a magnetic field.

In conclusion I wish to thank S. I. Syrovat-skii for valuable remarks.

[1] V. I. Veksler, Doklady Akad. Nauk SSSR **118**, 263 (1958), Soviet Phys. Doklady **3**, 84 (1958).

[2] V. C. A. Ferraro, J. Geophys. Research **57**, 15 (1952).

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