PHENOMENOLOGICAL DERIVATION OF THE EQUATIONS OF VORTEX MOTION IN He II

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A phenomenological derivation, based upon conservation laws, of the equations of vortex motion in He II is presented, with dissipative processes taken into account.

IN analyzing the oscillations of a stack of discs in rotating He II, Hall[1] introduced elastic forces for the vortex filaments in order to explain the experimental results. L. D. Landau advanced the idea that these forces may be derived phenomenologically with the aid, as usual, of conservation laws. Such a derivation is all the more to be desired in that it also permits the derivation, in a wholly rigorous manner, of the dissipative terms, which, Hall[1] felt could not be expressed in closed form through the velocity of motion of the liquid.*

A paper has recently been published by Mamaladze and Matinyan,[3] in which the problem of the oscillations of a single disc in rotating He II has been solved. The equations used therein, however, are incomplete, in that for $\omega = \text{curl}\, \mathbf{v}_s \neq 0$ the mutual friction forces no longer depend upon the difference $\mathbf{v}_n - \mathbf{v}_s$ alone, as assumed in reference 3.

In the present paper a complete system of hydrodynamic equations is derived for He II in which vortical motion of the superfluid component is taking place, with dissipative processes taken into account.

Let us consider He II in the case of a non-vanishing curl of the superfluid velocity. The fundamental distinction between the motion under consideration and the curl-free case lies in the dependence of the internal energy ϵ of the liquid upon the absolute value of the curl of the velocity. This dependence may be expressed in the differential form:

$$\delta\varepsilon = \lambda\delta\omega, \qquad \omega = |\text{rot}\, \mathbf{v}_s|. \qquad (1)^\dagger$$

The coefficient λ, as follows from a microscopic treatment of the vortex filaments, is given by

$$\lambda = \rho_s \frac{h}{m} \ln \frac{R}{a},$$

where ρ_s is the superfluid component density, m is the mass of the He^3 atom, and R/a is the ratio of the distance between vortices to the effective radius of a vortex.

We shall henceforth follow the standard method, proceeding from the conservation laws.

In the momentum flux tensor Π_{ik} and in the energy flux $\mathbf{Q}$ there appear additional terms which we shall designate π_{ik} and $\mathbf{q}$. The mass current vector $\mathbf{j}$, which equals the momentum per unit volume of the liquid, can, as usual, be defined in such a way that the continuity equation retains its form. Taking this into account, the equations for conservation of mass, energy, and momentum will be

$$\partial\rho/\partial t + \text{div}\, \mathbf{j} = 0, \qquad (2)$$

$$\partial E/\partial t + \text{div}\, (\mathbf{Q}_0 + \mathbf{q}) = 0, \qquad (3)$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial}{\partial x_k}(\Pi_{ik}^0 + \pi_{ik}) = 0, \qquad (4)$$

where ρ is the density of the liquid, and E is the energy per unit volume of the He II. The latter can, with the aid of a Galilean transformation, be expressed as an internal energy ϵ of the liquid in a reference system moving with the superfluid component velocity $\mathbf{v}_s$:

$$E = \tfrac{1}{2}\rho v_s^2 + \mathbf{p}\mathbf{v}_s + \varepsilon, \qquad (5)^*$$

where $\mathbf{p}$ is the momentum of the liquid per unit volume in this same system. The total momentum per unit volume, in the stationary coordinate system, is also expressed in terms of $\mathbf{p}$:

$$\mathbf{j} = \mathbf{p} + \rho\mathbf{v}_s. \qquad (6)$$

The internal energy ϵ obeys a thermodynamic identity which must take into account the previously-indicated (1) change in the energy of the liquid due to the vortex motion:

$$d\varepsilon = TdS + \mu d\rho + (\mathbf{v}_n - \mathbf{v}_s, d\mathbf{p}) + \lambda\, d\omega, \qquad (7)$$

*While the present paper was being prepared for publication, there appeared a review by Hall,[2] in which he has derived, by less general means, an expression analogous to ours for the mutual friction force. His final result, however, differs from ours, evidently as a result of a mathematical error of his.

†rot = curl.

*$\mathbf{pv}$ or $(\mathbf{p}, \mathbf{v}) = \mathbf{p} \cdot \mathbf{v}$.

where S and T are the entropy and temperature, μ is the chemical potential, and $\mathbf{v}_n$ is the normal component velocity.

We shall take the undisturbed momentum flux tensor and energy flux in the form[4]:

$$\mathbf{Q}_0 = (\mu + \tfrac{1}{2} v_s^2)\,\mathbf{j} + ST\mathbf{v}_n + \mathbf{v}_n(\mathbf{v}_n\mathbf{p}), \tag{8}$$

$$\Pi_{ik}^0 = \rho v_{si} v_{sk} + v_{si} p_k + v_{nk} p_i + p\delta_{ik}, \tag{9}$$

where p is the pressure:

$$p = -\varepsilon + TS + \mu\rho + (\mathbf{v}_n - \mathbf{v}_s,\ \mathbf{p}). \tag{10}$$

We thus include all of the dissipative processes in $\mathbf{q}$ and π_{ik}.

In addition to the conservation equations (2) — (4), the system of hydronamic equations includes the equation of motion of the superfluid component and the entropy growth equation. Taking into account the additional terms arising from the vortex motion, we have

$$\partial \mathbf{v}_s / \partial t + (\mathbf{v}_s \nabla)\,\mathbf{v}_s + \nabla\mu = \mathbf{f}, \tag{11}$$

$$\partial S/\partial t + \operatorname{div} S\mathbf{v}_n = R/T, \tag{12}$$

where the quantities $\mathbf{f}$ and R are to be determined (R is the dissipative function).

In order to determine the form of the unknown functions let us differentiate the left- and right-hand parts of (5) with respect to time. Substituting the time derivatives from (2), (4), and (11), we obtain

$$\begin{aligned}\dot E = &-\operatorname{div} Q_0 - \operatorname{div}(\pi\mathbf{v}_n) + T(\dot S + \operatorname{div} S\,\mathbf{v}_n)\\ &+ \lambda\dot\omega + \lambda v_{ni}\,\partial\omega/\partial x_i + \pi_{ik}\,\partial v_{ni}/\partial x_k\\ &+ (\mathbf{j} - \rho\mathbf{v}_n,\ \mathbf{f} + [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s]),\end{aligned} \tag{13}$$

where

$$(\pi\mathbf{v}_n)_i = \pi_{ik}\, v_{nk}.$$

Introducing the notation $\boldsymbol{\nu} = \boldsymbol{\omega}/\omega$ and substituting into (13) the expression

$$\lambda\dot\omega = \lambda\boldsymbol{\nu}\operatorname{rot}\dot{\mathbf{v}}_s = \lambda\boldsymbol{\nu}\operatorname{rot}\{\mathbf{f} + [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s]\} - \lambda\boldsymbol{\nu}\operatorname{rot}[\boldsymbol{\omega}\mathbf{v}_n], \tag{14*}$$

we obtain, after certain transformations which separate out the terms having the form of divergences:

$$\begin{aligned}\dot E &+ \operatorname{div}\{\mathbf{Q}_0 + (\pi\mathbf{v}_n) + \lambda[\boldsymbol{\nu},\ \mathbf{f} + [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s]]\}\\ &= T(\dot S + \operatorname{div} S\mathbf{v}_n) + (\pi_{ik} - \lambda\omega\delta_{ik} + \lambda\omega_i\omega_k/\omega)\,\partial v_{ni}/\partial x_k\\ &+ (\mathbf{f} + [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s],\ \mathbf{j} - \rho\mathbf{v}_n + \operatorname{rot}\lambda\boldsymbol{\nu}).\end{aligned} \tag{15}$$

Comparing this expression with the equations for conservation of energy (3) and entropy growth (12), we find

$$\mathbf{q} = (\pi\mathbf{v}_n) + \lambda[\boldsymbol{\nu},\ \mathbf{f} + [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s]], \tag{16}$$

$$\begin{aligned}R = &-(\pi_{ik} - \lambda\omega\delta_{ik} + \lambda\omega_i\omega_k/\omega)\,\partial v_{ni}/\partial x_k\\ &-(\mathbf{f} + [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s],\ \mathbf{j} - \rho\mathbf{v}_n + \operatorname{rot}\lambda\boldsymbol{\nu}).\end{aligned} \tag{17}$$

Requiring that the dissipative function be a positive definite quadratic form, we obtain from (17) the most general expressions for the vector terms

$$\begin{aligned}\mathbf{f} = &-[\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s] + \alpha[\boldsymbol{\omega},\ \mathbf{j} - \rho\mathbf{v}_n + \operatorname{rot}\lambda\boldsymbol{\nu}]\\ &+ \beta[\boldsymbol{\nu}[\boldsymbol{\omega},\ \mathbf{j} - \rho\mathbf{v}_n + \operatorname{rot}\lambda\boldsymbol{\nu}]] - \gamma\boldsymbol{\nu}(\boldsymbol{\omega},\mathbf{j} - \rho\mathbf{v}_n + \operatorname{rot}\lambda\boldsymbol{\nu})\end{aligned} \tag{18}$$

($\beta,\ \gamma \geq 0$, as a consequence of the condition $R \geq 0$), and for the tensor terms

$$\pi_{ik} = \lambda\omega\delta_{ik} - \lambda\omega_i\,\omega_k/\omega + \tau_{ik}. \tag{19}$$

τ_{ik} appears as a viscous stress tensor and must be expressed in general form with the aid of the viscosity tensor η_{iklm}:

$$\tau_{ik} = \eta_{iklm}\,\partial v_{nl}/\partial x_m. \tag{20}$$

When vortical motion takes place in the liquid, with $\boldsymbol{\omega}$ assuming a specific orientation, the viscosity tensor may in principle contain additional terms, as compared with the vortex-free case. In precisely this way the momentum of relative motion $\mathbf{p}$ - which in vortex-free flow is directed, due to the isotropy of the liquid, along the relative velocity

$$\mathbf{p} = \rho_n(\mathbf{v}_n - \mathbf{v}_s), \tag{21}$$

where ρ_n is the normal density - may now have as well a component along the second, distinct direction $\boldsymbol{\omega}$.

In principle, transfer of entropy and momentum by the vortices is also possible.

All of these phenomena, however, are quadratic effects and are extremely small. In particular, as in the absence of vortical motion, the tensor η_{iklm} reduces to the coefficients of first and second viscosity (which are independent of ω). The thermal conductivity coefficient also remains unchanged, as do the dissipative gradient terms in the superfluidity equation (in our derivation, $\mathbf{f}$ is determined accurate to the gradient terms).

Substituting into the right-hand part of (18) the expression

$$\mathbf{j} - \rho\mathbf{v}_n = \mathbf{p} + \rho(\mathbf{v}_s - \mathbf{v}_n) = -\rho_s(\mathbf{v}_n - \mathbf{v}_s), \tag{22}$$

which is obtained with the aid of Eq. (21), we may transform $\mathbf{f}$ into the form

$$\begin{aligned}\mathbf{f} = &-\rho_s^{-1}[\boldsymbol{\omega}\operatorname{rot}\lambda\boldsymbol{\nu}] - (1 + \alpha\rho_s)[\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s - \rho_s^{-1}\operatorname{rot}\lambda\boldsymbol{\nu}]\\ &-\beta\rho_s[\boldsymbol{\nu}[\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s - \rho_s^{-1}\operatorname{rot}\lambda\boldsymbol{\nu}]]\\ &+\gamma\rho_s\boldsymbol{\nu}(\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s - \rho_s^{-1}\operatorname{rot}\lambda\boldsymbol{\nu}).\end{aligned} \tag{23}$$

Let us now write out the superfluidity equation

*$[\boldsymbol{\omega}\mathbf{v}_n] = \boldsymbol{\omega} \times \mathbf{v}_n$; $[\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s] = \boldsymbol{\omega} \times (\mathbf{v}_n - \mathbf{v}_s)$.

$$\partial \mathbf{v}_s/\partial t + (\mathbf{v}_s \nabla)\mathbf{v}_s + \nabla\mu = -\rho_s^{-1}[\boldsymbol{\omega}\,\mathrm{rot}\,\lambda\boldsymbol{\nu}]$$
$$-\beta'\rho_s[\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s - \rho_s^{-1}\mathrm{rot}\,\lambda\boldsymbol{\nu}]$$
$$-\beta\rho_s[\boldsymbol{\nu}[\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s - \rho_s^{-1}\mathrm{rot}\,\lambda\boldsymbol{\nu}]]$$
$$+\gamma\rho_s\boldsymbol{\nu}(\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s - \rho_s^{-1}\mathrm{rot}\,\lambda\boldsymbol{\nu}), \qquad (24)$$

where $\beta'\rho_s$ represents $1 + \alpha\rho_s$. The first term on the right coincides with that derived by Hall in reference 1. The remaining three terms describe the mutual friction force. For the case of straight-line vortices $\nu \equiv \omega/\omega$ = const., from which it is easy to derive the relation between the coefficients β and β' and the coefficients of Hall and Vinen:

$$\beta = \tfrac{1}{2}B\rho_n/\rho\rho_s, \quad \beta' = \tfrac{1}{2}B'\rho_n/\rho\rho_s.$$

In the case of transverse deflections of the vortices, the mutual friction includes additional terms in curl $\lambda\nu$. Under certain conditions these may play a predominant role.

The term having the coefficient γ represents a longitudinal (along ω) mutual friction force which, it appears, may arise when the direction of the individual vortex filaments deviates from the direction of the mean curl of the velocity ω; for example, in the presence of thermal oscillations. In view of the smallness of these effects, however, the coefficient γ is evidently extremely small by comparison with β and β'. The increments to the momentum flux tensor resulting from the vortex motion have the form

$$\tilde{\pi}_{ik} = \lambda\omega\delta_{ik} - \lambda\omega_i\omega_k/\omega \qquad (25)$$

and lead to renormalization of the pressure to the value $\lambda\omega$, and to the appearance of a term $-\lambda\omega_i\omega_k/\omega$, which may be interpreted as the vortex filament tension.

Let us now consider the change in the energy flux. Substituting into Eq. (16) $\tilde{\pi}_{ik}$ from (25) we obtain

$$\mathbf{q} = \lambda[\boldsymbol{\nu}, \mathbf{f} + [\mathbf{v}_s\boldsymbol{\omega}]]. \qquad (26)$$

This expression is especially easy to interpret in the absence of any longitudinal component of $\mathbf{f}(\gamma = 0)$. In this case we obtain from (24), taking the curl of both sides

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \mathrm{rot}\{\mathbf{f} + [\mathbf{v}_s\boldsymbol{\omega}]\} = \mathrm{rot}[\mathbf{v}_L\boldsymbol{\omega}], \qquad (27)$$

which has the form of a transport equation for the vector ω, with a velocity

$$\mathbf{v}_L = \mathbf{v}_s - \frac{1}{\rho_s}\mathrm{rot}\,\lambda\boldsymbol{\nu} + \beta'\rho_s(\mathbf{v}_n - \mathbf{v}_s - \mathrm{rot}\,\lambda\boldsymbol{\nu})$$
$$-\beta\rho_s\left[\boldsymbol{\nu};\ \mathbf{v}_n - \mathbf{v}_s - \frac{1}{\rho_s}\mathrm{rot}\,\lambda\boldsymbol{\nu}\right],$$

This may be called the velocity of movement of the vortex filaments (and is accurate to terms parallel to the vector ω, which is adequate in the absence of longitudinal friction).

Thus, as follows from (27), the energy flux may be written in the form

$$\mathbf{q} = \omega\frac{\partial e}{\partial\omega}[\boldsymbol{\nu}[\mathbf{v}_L, \boldsymbol{\nu}]],$$

which represents energy transport by the vortices in a direction perpendicular to ω.

Finally, we shall consider the boundary conditions which must be obeyed by the superfluid component at the surface of a solid and at a free surface as the normal component velocity falls to zero. We shall limit our treatment to the vortices terminating at the surface.

It is clear that in the case of a free surface the tangential components of the tension in the liquid must be zero; i.e., $(\pi\mathrm{N})_t = 0$, whence $\omega_N \cdot \omega_t = 0$. The vortices are thus perpendicular to the free surface of the liquid (N is the normal vector).

In the case of a very rough surface the velocity of the vortices at the surface must coincide with that of the surface itself.

In the event of vortex slippage, however, intensive energy dissipation must take place, as observed by Hall.[1]

The mechanical energy of the vortex motion dissipated at the surface is numerically equal to the difference between the energy flux directed toward the surface and the mechanical energy actually transferred to the surface, the latter being equal to the work done by the tension forces as the wall moves at a velocity $\mathbf{u} = \mathbf{v}_n|_{\mathrm{surface}}$;

$$-\dot{E}_{\mathrm{mech}} = \mathbf{qN} - (\tilde{\pi}\mathbf{u})\,\mathbf{N}. \qquad (28)$$

Here, the vector $(\pi\mathbf{u})$ is determined in accordance with (13), and N is the normal vector external to the liquid.

Substituting q and $\tilde{\pi}_{ik}$ from (25) and (26), after interchanging the order of the factors in the mixed vector product

$$-\dot{E}_{\mathrm{mech}} = \lambda(\mathbf{f} + [\mathbf{v}_s - \mathbf{u}, \boldsymbol{\omega}],\ [\mathbf{N}\boldsymbol{\nu}]), \qquad (29)$$

from which it follows that

$$\mathbf{f} + [\mathbf{v}_s - \mathbf{u},\ \boldsymbol{\omega}]|_{\mathrm{sur}} = \zeta[\mathbf{N}\boldsymbol{\omega}] + \zeta'[[\mathbf{N}\boldsymbol{\nu}]\boldsymbol{\omega}]. \qquad (30)$$

From the qualitative microscopic treatment we obtain the values for the coefficients ζ and ζ':

$$\zeta/B \approx \zeta'/B' \approx \rho_n h/\rho\, md, \qquad (31)$$

where d is the mean size of the surface irregularities.

The boundary condition (30) may be rewritten in terms of the vortex velocity (for $\gamma = 0$):

$$\mathbf{v}_L - \mathbf{u} = \zeta[\boldsymbol{\nu}[\mathbf{N}\boldsymbol{\nu}]] + \zeta'[\mathbf{N}\boldsymbol{\nu}], \qquad (32)$$

which for $\zeta' = 0$ agrees with the boundary condition employed by Hall.[1] The transitions $\zeta, \zeta' \to 0$ and $\zeta, \zeta' \to \infty$ correspond to transitions to an absolutely rough surface and to a free surface, respectively.

We shall now write down the complete system of equations for an incompressible fluid. Let us determine the μ appearing in the superfluidity equation. Making use of the thermodynamic identity (7) and the definition of p (10) we find, if we take into account the fact that λ is weakly dependent upon ω,

$$\mu = \mu(p, T) - (\rho_n/2p)(\mathbf{v}_n - \mathbf{v}_s)^2 + \lambda\omega/\rho.$$

Moreover, introducing the renormalized pressure $p_0 = p + \lambda\omega$, as follows from (25), we have, finally

$$\mu = \mu(p_0 - \lambda\omega, T) + \frac{\lambda\omega}{\rho} - \frac{\rho_n}{2\rho}(\mathbf{v}_n - \mathbf{v}_s)^2$$

$$\approx \mu(p_0, T) - \frac{\rho_n}{2\rho}(\mathbf{v}_n - \mathbf{v}_s)^2. \quad (33)$$

The full system of equations of motion for an incompressible liquid ($\operatorname{div}\mathbf{v}_n = \operatorname{div}\mathbf{v}_s = 0$) consists of the superfluidity equation (24), in which $\nabla\mu$ must be replaced by $p_s = \rho_s p/\rho$, and the equation of motion of the normal component (obtained from the momentum conservation equation):

$$\rho_n[\partial\mathbf{v}_n/\partial t + (\mathbf{v}_n\nabla)\mathbf{v}_n] = -\nabla p_n + \eta\Delta\mathbf{v}_n$$

$$-\tfrac{1}{2}B\rho_s\rho_n\rho^{-1}[\boldsymbol{\omega}[\boldsymbol{\nu}, \mathbf{v}_n - \mathbf{v}_s - \lambda\rho_s^{-1}\operatorname{rot}\boldsymbol{\nu}]]$$

$$-\tfrac{1}{2}B'\rho_s\rho_n\omega^{-1}[\boldsymbol{\omega}, \mathbf{v}_n - \mathbf{v}_s - \lambda\rho_s^{-1}\operatorname{rot}\boldsymbol{\nu}],$$

$$p_n = p\rho_n/\rho, \quad (34)$$

under the boundary conditions

$$\mathbf{v}_n|_{\text{sur}} = \mathbf{u}_{\text{sur}}, \qquad (\mathbf{v}_s - \mathbf{u}, \mathbf{N})_{\text{sur}} = 0,$$

$$\mathbf{f} + [\mathbf{u} - \mathbf{v}_s, \boldsymbol{\omega}] = \zeta[\mathbf{N}\boldsymbol{\omega}] + \zeta'[[\mathbf{N}\boldsymbol{\nu}]\boldsymbol{\omega}]. \quad (35)$$

It is possible, using the hydrodynamic equations thus obtained, to compute the moment of the forces acting upon a disc oscillating in rotating He II.

If the frequency of the disc oscillations is represented by Ω, and the angular frequency of rotation of the liquid by ω_0, then, with accuracy out to $B\rho_n\omega_0/\rho\Omega$, the motion of the normal component will not influence the motion of the superfluid. It also turns out that, to the same degree of accuracy, the change in the moment of the forces acting on the disc on the part of the normal component $\eta B/\lambda \sim 1/30$ of the moment of the forces due to vortex tension. We shall now present the expression for the moment of the forces acting on a disc at a distance l below the free surface of the liquid (these calculations were carried out to the accuracy indicated above):

$$\Delta M = M_s = \alpha_0\pi R^4\lambda\omega_0\left(\frac{N_1}{1-(\zeta'+i\zeta)N_1/\Omega} + \frac{N_2}{1+(\zeta'-i\zeta)N_2/\Omega}\right), \quad (36)^*$$

where

$$N_i = k_i \operatorname{tg} k_i l \quad (i = 1, 2),$$

$$k_1 = \sqrt{\frac{\Omega - 2\omega_0}{\lambda}}\left(1 + i\frac{B\rho_n}{4\rho}\right),$$

$$k_2 = i\sqrt{\frac{\Omega + 2\omega_0}{\lambda}}\left(1 + i\frac{B\rho_n}{4\rho}\right),$$

α_0 is the amplitude of disc oscillations, and R is the radius of the disc. In this approximation, Eq. (36) agrees, for the condition $l \to \infty$ with that obtained by Mamaladze and Matinyan[3] (it would be improper to take into consideration the higher degree of precision indicated in reference 3, in view of the incompleteness of the equations used therein).

In conclusion, the authors express their deep gratitude to Academician L. D. Landau for his consideration of the problem under review and for his valued advice.

[1] H. E. Hall, Proc. Roy. Soc. **245**, 546 (1958).
[2] H. E. Hall, Adv. in Phys. **9**, 89 (1960).
[3] Yu. G. Mamaladze and S. G. Matinyan, JETP **38**, 184 (1960), Soviet Phys. JETP **11**, 134 (1960).
[4] I. M. Khalatnikov, JETP **23**, 265 (1952).

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*tg = tan.