ABSORPTION OF ULTRASOUND IN AN ANISOTROPIC SUPERCONDUCTOR

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Anisotropy of superconductors is taken into account within the framework of the Gor'kov method.[1] The electron-phonon interaction is not considered small, but the smallness of the sound velocity in comparison with the Fermi velocity is explicitly taken into account. It is shown that the low-temperature region $T \ll T_c$ is divided into two regions in which the behavior of the sound attenuation as a function of direction and temperature is significantly different. Sound absorption exhibits some peculiarities at low temperatures in semiconductors possessing Fermi surfaces of the "corrugated" type. The problem of the possibility of determining the energy gap from data on the attenuation of ultrasound is analyzed.

1. INTRODUCTION

RECENTLY several works have appeared which are devoted to the study of the effects of anisotropy in superconductors. There is first the theoretical work of Khalatnikov,[2] which is devoted to the anisotropy of heat conduction, and the experimental researches on the measurement of heat capacity,[3] heat conduction[4] and ultrasonic absorption.[5,6]

The natural purpose of the experimental researches was the determination of the energy gap Δ as a function of direction. However, as theory shows,[2] at temperatures T much less than the transition temperature T_c, the thermal conductivity is determined only by the minimum value of the energy gap, while a significant anisotropy in the thermal conductivity arises only in the case in which there are two minima for Δ (in opposite directions). Therefore, information on Δ obtained from experimental data on thermal conductivity is very limited.

So far as research on the measurement of ultrasonic absorption is concerned,[5,6] the authors therein, in their treatment of the experimental data, made use of the formula for the ratio α_s/α_n, introduced by Bardeen, Cooper and Schrieffer (denoted below as BCS) for the isotropic case (α_s, α_n — absorption coefficients of ultrasound in the superconducting and normal states, respectively):

$$\alpha_s/\alpha_n = 2f(\beta\Delta), \qquad f(x) = (e^x + 1)^{-1}, \qquad \beta = 1/T \,. \quad (1)$$

As will be shown in our work, Eq. (1) is not valid in the anisotropic case. The quantity $f(\beta\Delta)$ appears in the correct formula; it is averaged with some weight of very complicated form along the line (on the Fermi surface) on which the Fermi velocity is perpendicular to the direction of sound propagation. Therefore, experiments on the ultrasonic attenuation generally give incomplete information on the function $\Delta(\mathbf{n})$, although the results are much more fruitful than the measurements of thermal conductivity.

Moreover, Eq. (1) at low temperatures does not reflect a number of peculiarities which are properties of the anisotropic superconductors. Although the ultrasonic absorption in a given direction actually falls off exponentially with temperature, it is shown that the exponent, generally speaking, depends on the direction. At very low temperatures (see Sec. 4), the exponent does not depend on the direction but then the absorption has anomalies along certain directions.

Finally, the approximation made in the work of BCS (the smallness of the interaction constant between electrons) is found to be insufficient for the correct description of sound absorption in the region of low temperatures. In this connection, we shall construct a theory in which the quantity c/v_F is used as a small parameter (c — sound velocity, v_F — velocity of the electron on the Fermi surface).

2. THE ELECTRON-PHONON INTERACTION GREEN'S FUNCTION

We shall limit ourselves to consideration of the interaction of electrons with phonons, which we shall describe by a Hamiltonian of the Fröhlich type:[7]

$$H = \sum_{s,\mathbf{p},\mathbf{q}} \gamma_s(\mathbf{p},\mathbf{q})\, a_{\mathbf{p}}^+ a_{\mathbf{p}+\mathbf{q}} b_{s\mathbf{q}}^+ + \text{Herm. conj.} \quad (2)$$

where a_p^+, a_p are the creation and annihilation operators of electrons and holes with momentum **p**: b_{sq}^+ is the creation operator of a phonon with momentum **q** and polarization s ($s = 1, 2, 3$); $\gamma_s(\mathbf{p}, \mathbf{q})$ is the matrix element of the energy of interaction. For sufficiently small $\mathbf{q}$ ($q \ll p_0 \sim \pi/a$, where a is the lattice constant) $\gamma_s(\mathbf{p}, \mathbf{q})$ can be represented in the form*

$$\gamma_s(\mathbf{p}, \mathbf{q}) = \beta_s(\mathbf{q})\, g_0(\mathbf{p}), \quad \beta_s^2(\mathbf{q}) = \pi^2 \lambda_s(\mathbf{n})\, \omega_s^0(\mathbf{q}) / p_0, \quad \mathbf{n} = \mathbf{q}/q. \tag{3}$$

Here, $\omega_s^0(\mathbf{q})$ — frequency of a phonon with momentum **q** and polarization s; $\lambda_s(\mathbf{n})$ — nondimensional function of the interaction. As usual, $\omega_s^0(\mathbf{q}) = c_s(\mathbf{n})q$, where $c_s(\mathbf{n})$ — sound velocity, $g_0(\mathbf{p})$ — some dimensionless function which describes the anisotropy of the interaction.

Following Matsubara,[8] we introduce the Green's function (more precisely, tensor) of the phonons:

$$D_{ss'}(\mathbf{q}, \tau - \tau') = i \langle T_\tau b_s(\mathbf{q}, \tau)\, b_s^+(\mathbf{q}, \tau') \rangle\, \beta_s(\mathbf{p})\, \beta_{s'}(\mathbf{q}), \tag{4}$$

where $\langle A \rangle$ denotes the trace of $e^{-\beta(H-\mu N)}A$ and

$$b_s(\mathbf{p}, \tau) = e^{\tau H} b_s(\mathbf{p})\, e^{-\tau H}, \qquad b_s^+(\mathbf{p}, \tau) = e^{\tau H} b_s^+(\mathbf{p})\, e^{-\tau H}.$$

As Gor'kov has shown,[1] in the case of a superconductor the electrons are described by two Green's functions G and F, which we shall define as:

$$G(\mathbf{p}, \tau - \tau') = i \langle T_\tau a(\mathbf{p}, \tau)\, a^+(\mathbf{p}, \tau') \rangle\, g_0(\mathbf{p}), \tag{5}$$

$$F(\mathbf{p}, \tau - \tau') = \langle N | e^{\beta(H-\mu N)} T a^+(\mathbf{p}, \tau)\, a^+(-\mathbf{p}, \tau') | N + 2 \rangle\, g_0(\mathbf{p}). \tag{6}$$

In Eq. (6), $\langle N|$, $|N+2\rangle$ are the microcanonical states with N and N + 2 particles, corresponding to the given temperature.

In what follows we shall transform from the "time" representation of the functions, D, G, F [Eqs. (4), (5), (6)] to the "representation of imaginary frequencies," developed in references 9 and 10. We shall use for construction of the functions $D_{ss'}(\mathbf{q}, i\omega_n)$, $G(\mathbf{p}, i\eta_n)$, and $F(\mathbf{p}, i\eta_n)$ (where $\omega_n = 2n\pi T$, $\eta_n = (2n+1)\pi T$), a diagram technique in which the "zero" Green's functions correspond to the lines:

$$D_{ss'}^0 = \delta_{ss'} D_s, \quad D_s(\mathbf{q}, i\omega_n) = 2\beta_s^2(\mathbf{q})\, \omega_s^0(\mathbf{q}) / [\omega_s^{02}(\mathbf{q}) + \omega_n^2], \tag{7}$$

$$G^0(\mathbf{p}, i\eta_n) = g_0(\mathbf{p}) / (\xi_\mathbf{p}^0 - i\eta_n), \tag{8}$$

$$F^0(\mathbf{p}, i\eta_n) = 0, \tag{9}$$

where, as usual,

$$\xi_\mathbf{p} = v_F^0(\mathbf{n})(\mathbf{p} - \mathbf{p}_F). \tag{10}$$

Here v_F is considered to be dependent on the direction; the Fermi surface $\xi_\mathbf{p} = 0$ represents some arbitrary surface in momentum space. For a given choice of "zero" Green's functions, the elementary vertex in the diagram corresponds to the factor $-i$.

We introduce the polarization operator Π in the usual way:

$$D_{ss'} = D_{ss'}^0 + \sum_{r, r'} D_{sr}^0 \Pi_{rr'} D_{r's}. \tag{11}$$

It is obvious that in our definition of the Green's functions (4) and (7) — (9) the operator $\Pi_{rr'}$ does not depend on the indices r, r':

$$\Pi_{rr'} = \Pi. \tag{12}$$

Substituting (12) in (11), and solving the complete set of equations relative to $D_{ss'}$, we get

$$D_{ss'} = \delta_{ss'} D_s + D_s D_{s'} \Pi / (1 - D\Pi), \qquad D = \sum_s D_s. \tag{13}$$

The frequency and damping of the acoustic vibrations are determined by the poles $\omega = \omega_s(\mathbf{q})$ of the functions $D_{ss'}(\mathbf{q}, \omega)$, which are analytically continued with discrete imaginary values $\omega = i\omega_n$ in the complex plane ω. We recall that $D_{ss'}(\mathbf{q}, \omega)$ should be analytic in the upper half-plane of ω; the poles $\omega_s(\mathbf{q})$ should located in the lower half-plane close to the real axis. Obviously, the poles $D_{ss'}(\mathbf{q}, \omega)$ coincide with the roots of the equation

$$1 - D\Pi = 0. \tag{14}$$

We emphasize that the poles D_s are not the poles $D_{ss'}$ (for $\Pi \neq 0$).

The Dyson equations for the functions D, G, F can be found in the work of the author.[11] In the equations described there [Eqs. (3) — (8)] obvious changes should be made corresponding to the transition from integration over the frequency to summation, and also corresponding to summation over the polarizations.

It can be shown (compare reference 12) that in the regions of interest to us ($\omega \sim cq$) the complete vertex part Γ_1 can be replaced by unity, with accuracy up to a small quantity $\sim c/v_F$, and Γ_2 by zero.

This makes it possible to apply the method developed by Migdal for the solution of the Dyson equations[12] (see also reference 13). However, we have a direct interest only in the attenuation of the ultrasound. Therefore we shall not be concerned here with the exact solution of the Dyson equations, but shall note only some properties of G and F, which are necessary in what follows. We write down the Dyson equations for G and F:

*It would have been more accurate to write $\gamma_s(\mathbf{p}, \mathbf{q})$ in the form of a sum of products of the type (3), which would have greatly complicated the calculations without changing the results.

FIG. 1

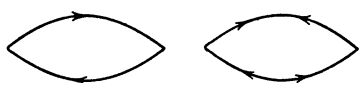

FIG. 2

$$G(p) = G_0(p) + [G_0(p)\Sigma_1(p)G(p) - G_0(p)\Sigma_2(p)F(p)]g_0^{-1}(\mathbf{p}), \quad (15)$$

$$F(p) = [G_0(-p)\Sigma_1(-p)F(p) + G_0(-p)\Sigma_2(p)G(p)]g_0^{-1}(\mathbf{p}), \quad (16)$$

where p is the 4-vector $(\mathbf{p}, i\eta_n)$. The operators of the "self energy" Σ_1, Σ_2 in the given approximation are determined by the diagrams of Fig. 1, where the smooth lines correspond to the complete Green's functions.

First we note that the values $\xi_p \gg \Delta$ play a role in Σ_1, and therefore $\Sigma_1(p)$ has practically the same value as in a normal metal. Furthermore, for values of T of interest to us (below the degeneracy temperature) the function $\Sigma_1(p)$ is almost independent of T, and we can replace it by the value at zero temperature. We now introduce the function $G_M(p)$

$$G_M(p) = G_0(p)[1 - g_0^{-1}(\mathbf{p})G_0(p)\Sigma_1(p)]^{-1}. \quad (17)$$

The function $G_M(p)$ is identical with that found by Migdal[12] for the normal metal (naturally, with account of anisotropy and replacement of η for $i\eta_n$). In reference 12 it was shown (the proof and result are only slightly changed upon consideration of anisotropy) that $G_M^{-1}(p)$ has the form

$$G_M^{-1}(p) = (\xi_\mathbf{p} - \varphi(\mathbf{n};\eta))g_0^{-1}(\mathbf{p}), \quad \mathbf{n} = \mathbf{p}/p, \quad (18)$$

where $\varphi(\mathbf{n}, \eta)$ is an odd function of η, while $\operatorname{Im}\varphi \ll \operatorname{Re}\varphi$ both for $\eta \ll \omega_0$ and for $\eta \gg \omega_0$ (ω_0 — Debye frequency) [the factor $g_0^{-1}(\mathbf{p})$ is, of course, absent in reference 12].

Solving the system (15), (16) with account of the definition (17), we obtain

$$G(\mathbf{p}, \eta) = \left(\frac{u_\mathbf{p}^{02}}{\varepsilon_\mathbf{p}^0 - \varphi(\mathbf{n}, \eta)} - \frac{v_\mathbf{p}^{02}}{\varepsilon_\mathbf{p}^0 + \varphi(\mathbf{n}, \eta)}\right) g_0(\mathbf{p}), \quad (19)$$

$$F(\mathbf{p}, \eta) = \frac{\Sigma_2}{2\varepsilon_\mathbf{p}^0}\left(\frac{1}{\varepsilon_\mathbf{p}^0 - \varphi(\mathbf{n}, \eta)} + \frac{1}{\varepsilon_\mathbf{p}^0 + \varphi(\mathbf{n}, \eta)}\right) g_0(\mathbf{p}), \quad (20)$$

where

$$u_\mathbf{p}^{02} = \tfrac{1}{2}(1 + \xi_\mathbf{p}^0/\varepsilon_\mathbf{p}^0), \quad v_\mathbf{p}^{02} = \tfrac{1}{2}(1 - \xi_\mathbf{p}^0/\varepsilon_\mathbf{p}^0), \quad \varepsilon_\mathbf{p}^{02} = \xi_\mathbf{p}^{02} + \Sigma_2^2, \quad (21)$$

while the dependence of Σ_2 on $|\mathbf{p}|$ and η can be neglected.

The energy of the electron in the superconductor is determined by the roots of the equation

$$\varphi(\eta) = \varepsilon_\mathbf{p}^0. \quad (22)$$

We denote the solution of this equation by $\epsilon_\mathbf{p}$. Equation (22) reduces to a certain renormalization of the quantities $\xi_\mathbf{p}$ and Σ_2 (we emphasize that the renormalization of $\xi_\mathbf{p}$ and Σ_2 is the same). Therefore, we can write Eqs. (19), (20) in the form

$$G(\mathbf{p}, \eta) = \left(\frac{u_\mathbf{p}^2}{\varepsilon_\mathbf{p} - \eta} - \frac{v_\mathbf{p}^2}{\varepsilon_\mathbf{p} + \eta}\right) g(\mathbf{p}) + A(\mathbf{p}, \eta),$$

$$F(\mathbf{p}, \eta) = \frac{\Delta(\mathbf{p})}{2\varepsilon_\mathbf{p}}\left(\frac{1}{\varepsilon_\mathbf{p} - \eta} + \frac{1}{\varepsilon_\mathbf{p} + \eta}\right) g(\mathbf{p}) + B(\mathbf{p}, \eta). \quad (23)$$

The functions without the index refer to the renormalized quantities; $\Delta(\mathbf{p})$ (the energy gap) is the renormalized Σ_2. The functions $A(\mathbf{p}, \eta)$, $B(\mathbf{p}, \eta)$ no longer have divergencies and are single-valued in the complex plane η with two cuts along the real axis $(\pm\Delta, \pm\infty)$.[11,13] However, in the approximation which we have made we have, for real η, $\operatorname{Im} A = \operatorname{Im} B = 0$, while $\operatorname{Re} A$ and $\operatorname{Re} B$ are single-valued functions of η.

3. THE ATTENUATION OF THE ULTRASOUND

We shall make a detailed study of the function $\Pi(\mathbf{p}, i\omega_n)$ which is defined by the diagram of Fig. 2:

$$\Pi(\mathbf{q}, i\omega_n) = \frac{T}{(2\pi)^3}\sum_{\eta_n}\int d^3p\,[G(p)G(p-q) - F(p)F(p-q)]. \quad (24)$$

It is convenient to go from summation over η_n to integration by means of

$$T\sum_{\eta_n}\cdots \to \frac{i}{4\pi}\int_\Gamma d\eta\, \operatorname{th}\frac{\beta\eta}{2}\cdots, \quad (25)^*$$

where Γ is a contour consisting of two straight lines parallel to the imaginary axis η. In integration over η, we deform the contour Γ so that there remain residues at the poles $\eta = \pm\epsilon_\mathbf{p}$ or integrals over the cuts. By virtue of the remark made in Sec. 2, the integrals over the cuts vanish in our approximation. Further, components appear of the type

$$\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\,g(\mathbf{p})\,u_\mathbf{p}^2 A(\mathbf{p} - \mathbf{q}, \varepsilon_\mathbf{p} - \omega). \quad (26)$$

The integration here is effectively carried out over the regions in which $p \sim p_0$, $\epsilon_p \sim \omega_0$. We shall be interested in the values $q \ll p_0$, $\omega \ll \omega_0$. Therefore, these components contribute a real constant quantity to Π. The remaining component has the form

*th = tanh.

$$\Pi'(\mathbf{q}, i\omega_n) = \frac{1}{2(2\pi)^3}\int d^3 p g^2(\mathbf{p}) \Big\{(f'-f)\Big(u^2u'^2+v^2v'^2-\frac{\Delta\Delta'}{2\varepsilon\varepsilon'}\Big) \times\Big(\frac{1}{\varepsilon-\varepsilon'-i\omega_n}+\frac{1}{\varepsilon-\varepsilon'+i\omega_n}\Big)+\Big(\operatorname{th}\frac{\beta\varepsilon}{2}+\operatorname{th}\frac{\beta\varepsilon'}{2}\Big) \times\Big(\frac{u^2v'^2+\Delta\Delta'/4\varepsilon\varepsilon'}{\varepsilon+\varepsilon'-i\omega_n}+\frac{v^2u'^2+\Delta\Delta'/4\varepsilon\varepsilon'}{\varepsilon+\varepsilon'+i\omega_n}\Big)\Big\}. \quad (27)$$

Here the unprimed quantities have the argument $\mathbf{p}$ and the primed ones the argument $\mathbf{p} - \mathbf{q}$. In order to go from the function (27), which is defined at the discrete values $\omega = i\omega_n$, to the function $\Pi'(\mathbf{q}, \omega)$, which is analytic in the upper half-plane of ω, it is necessary to replace $i\omega_n$ by $\omega + i\delta$ everywhere in (14). Then we find Re Π', Im Π' by assuming ω to be real. For the reason given in the study of (26) we can regard Re Π' as a constant. Im Π has the form

$$\operatorname{Im}\Pi=\frac{\pi}{2(2\pi)^3}\int d^3\mathbf{p}\, g^2\Big\{(f'-f)\Big(u^2u'^2+v^2v'^2-\frac{\Delta\Delta'}{2\varepsilon\varepsilon'}\Big)(\delta(\varepsilon-\varepsilon'-\omega) -\delta(\varepsilon-\varepsilon'+\omega))+\Big(\operatorname{th}\frac{\beta\varepsilon}{2}+\operatorname{th}\frac{\beta\varepsilon'}{2}\Big)\Big[\Big(u^2v'^2+\frac{\Delta\Delta'}{4\varepsilon\varepsilon'}\Big) \times\delta(\varepsilon+\varepsilon'-\omega)-\Big(u'^2v^2+\frac{\Delta\Delta'}{4\varepsilon\varepsilon'}\Big)\delta(\varepsilon+\varepsilon'+\omega)\Big]\Big\}. \quad (28)$$

The components in (28) which are proportional to $\delta(\epsilon + \epsilon' \pm \omega)$ are brought about by processes of decay of the phonon into an electron-hole pair, and only make a non-zero contribution to the damping of the phonons above the threshold[11] $\omega_{th} = 2\Delta$ ($\omega_{th} \sim 10^{11} - 10^{12}$ cps). We shall be interested in much lower frequencies for which the damping is determined by components with $\delta(\epsilon - \epsilon' \pm \omega)$.

We first note that both components are equal. Further, under our assumptions, we set $\Delta = \Delta'$, $u = u'$, $v = v'$, and $\epsilon = \epsilon'$ [of course, this is impossible to do in arguments of δ functions and in $f(\beta\epsilon')$]. Equation (28) then takes the form

$$\operatorname{Im}\Pi = \frac{1}{(2\pi)^2}\int d^3\mathbf{p}\, g^2(\mathbf{p})\frac{\xi^2}{\varepsilon^2}[f(\beta\varepsilon-\beta\omega) - f(\beta\varepsilon)]\;\delta(\varepsilon-\varepsilon'-\omega). \quad (29)$$

The functions $f(\beta\epsilon)$, $f(\beta\epsilon - \beta\omega)$ limit the integration over ξ in (29) to the region $\xi \sim \Delta$. We then make the additional assumption that $vq \ll \Delta$ (for real phonons, this corresponds to frequencies $\omega < 10^{10}$). Then, in the important region of integration, $\epsilon \gg vq$, and $\epsilon - \epsilon'$ can be put in the form

$$\varepsilon-\varepsilon' = vq\,\xi/\varepsilon. \quad (30)$$

Substituting (30) in (29), and neglecting the small quantities $\sim c/v$, we get

$$\operatorname{Im}\Pi = \frac{1}{(2\pi)^2}\int d^3p\,\frac{\xi}{\varepsilon}(f(\beta\varepsilon-\beta\omega)-f(\beta\varepsilon))\,g^2\delta(vq). \quad (31)$$

We transform in the integral (31) to integration over ξ and over the surface ξ = const. Making use of the fact that the integration over ξ is limited to the region ξ = const, which is small in comparison with μ, we can transform the integral (31) in the following way:

$$\operatorname{Im}\Pi = \frac{2}{(2\pi)^2 q}\int\frac{d\sigma}{v_F}g^2(\mathbf{n})\,\delta(\cos\chi)\int_\Delta^\infty d\varepsilon\,(f(\beta\varepsilon-\beta\omega)-f(\beta\varepsilon)), \quad (32)$$

where χ is the angle between the direction of the Fermi velocity and the direction of propagation of the sound. If $\omega \ll T$ (this case exists in real experiments), then we obtain

$$\operatorname{Im}\Pi = \frac{2}{(2\pi)^2}\frac{\omega}{q}\int\frac{d\sigma}{v_F}f(\beta\Delta)\,g^2(\mathbf{n})\,\delta(\cos\chi). \quad (33)$$

We note that for real phonons ($\omega \sim cq$) the ratio Im Π/Re Π is of the order $(c/v)e^{-\beta\Delta}$, where we choose for Δ the minimum value of $\Delta(\mathbf{p})$ on the line $\cos\chi = 0$.

Equation (14), which determines the ultrasonic dispersion, has the form

$$\frac{\pi^2}{p_0}\sum_s\frac{\lambda_s(\mathbf{n})\,\omega_s^{02}(\mathbf{q})}{\omega_s^{02}(\mathbf{q})-\omega^2} = \frac{1}{\operatorname{Re}\Pi} - i\frac{\operatorname{Im}\Pi}{(\operatorname{Re}\Pi)^2}. \quad (34)$$

The first term on the right side of (34) reduces to the finite renormalization of the sound velocity, while the second corresponds to the damping. In view of its smallness, it is evident that the damping can be written in the form

$$\alpha_s(\mathbf{q}) = \operatorname{Im}\omega_s(\mathbf{q}) = \nu_s(\mathbf{n})\,\omega_s(\mathbf{q})\operatorname{Im}\Pi, \quad (35)$$

where $\nu_s(\mathbf{n})$ is some function of the direction that has not been measured experimentally.

Equations (35) and (33) show that the damping of the sound has a clearly expressed anisotropy.

We can eliminate the function $\nu_s(\mathbf{n})$ by considering the ratio of the damping of the sound in the superconductor (35) to the corresponding value $\alpha_n(\mathbf{q})$ in the normal metal, since the formula for $\alpha_n(\mathbf{q})$ is identical with (35) if we set $f(\beta\Delta) = 1/2$ in Eq. (33) for Im Π. The formula obtained by elimination of $\nu_s(\mathbf{n})$ is the analogue of Eq. (1) for the anisotropic case:

$$\frac{\alpha_s}{\alpha_n} = 2\langle f(\beta\Delta)\rangle_n,$$

$$\langle\varphi\rangle_n = \int\frac{d\sigma'}{v_F'}g^2(\mathbf{n}')\,\delta(\cos\chi)\,\varphi(\mathbf{n}')\Big/\int\frac{d\sigma'}{v_F'}g^2(\mathbf{n}')\,\delta(\cos\chi). \quad (36)$$

Unfortunately, a second unmeasurable function $g^2(\mathbf{n})$, which enters into (36), does not permit us to draw any conclusions about $\Delta(\mathbf{n})$.

4. ULTRASONIC ABSORPTION IN THE REGION OF LOW TEMPERATURES

Let us now consider the region of low temperatures $\beta\Delta \ll 1$. We can carry out integration in Eq. (29) over ξ, obtaining

$$\operatorname{Im}\Pi=\frac{\beta\omega c^2}{(2\pi)^2 q}\int\frac{d\sigma}{v_F^4}\frac{g^2\Delta}{|\cos^3\chi|}\exp\left[-\beta\Delta\left(1-\frac{c^2}{v_F^2\cos^2\chi}\right)^{-1/2}\right]. \quad (37)$$

Here the integration is not extended over the region $\cos\chi < c/v_F$. We transform to integration over the stereographic projection of the Fermi surface, where we use χ as the polar angle:

$$d\sigma = K^{-1}(\chi,\varphi)\, d(\cos\chi)\, d\varphi, \quad (38)$$

where $K(\chi, \varphi)$ is the Gaussian curvature of the Fermi surface. Let (χ_0, φ_0) be the point at which $\Delta(\chi, \varphi)$ has an absolute minimum, equal to Δ_0, and $\cos\chi_0 \sim 1$. Then (χ_0, φ_0) is a stationary point of the integrand of (37), the contribution of which I_0 in Im Π is determined by the equation

$$I_0=\frac{g^2}{4\pi\rho}\frac{\omega}{q}\frac{c^2}{Kv_F^4}\frac{1}{|\cos^3\chi_0|}\exp\left[-\beta\Delta_0\left(1-\frac{c^2}{v_F^2\cos^2\chi}\right)^{-1/2}\right],$$

$$\rho\Delta_0=\frac{\partial^2\Delta}{\partial(\cos\chi)^2}\frac{\partial^2\Delta}{\partial\varphi^2}-\left(\frac{\partial^2\Delta}{\partial(\cos\chi)\,\partial\varphi}\right)^2. \quad (39)$$

In Eq. (39), all the functions are taken at the point of the minimum (χ_0, φ_0).

Because of the smallness of c/v_F the region of small $\cos\chi$ can make a relatively large contribution to the integral (37). For the investigation of the contribution of I_1 in Im Π from this region, it is convenient to introduce the variable $u = c/v_F \times \cos\chi$. We have

$$I_1=\frac{\beta\omega}{4\pi^2 q}\int_0^{2\pi}d\varphi\int_{c/v_F}^{1}\frac{du}{Kv_F^2}g^2\Delta\exp\left[-\beta\Delta(1-u^2)^{-1/2}+\ln u\right]. \quad (40)$$

The equation determining the saddle point has the form

$$\frac{\beta\Delta u}{(1-u^2)^{3/2}}-\frac{1}{u}-\beta\frac{\partial\Delta}{\partial(\cos\chi)}\frac{c}{v_F u^2}(1-u^2)^{-1/2}=0. \quad (41)$$

We shall consider the last component on the right side of (41) to be small (below, we shall explain when this is valid). Then Eq. (41) has the solution $u = (\beta\Delta)^{-1/2}$. The condition under which the approximations we have made are valid has the form $(\beta\Delta)^{3/2} c/v_F \ll 1$. In this approximation, I_1 reduces to the form

$$I_1=\sqrt{\frac{\pi}{2e}}\frac{\omega}{4\pi^2 q}\int_0^{2\pi}d\varphi\,\frac{g^2}{Kv_F^2}e^{-\beta\Delta}. \quad (42)$$

Here all the functions are evaluated for $\cos\chi = 0$. The integration over φ in (42) can also be carried out in explicit form by making use of the large quantity $\beta\Delta$:

$$I_1=\frac{1}{4\pi\sqrt{2e\sigma}}\frac{\omega}{q}\frac{g^2}{Kv_F^2}(\beta\Delta_0)^{-1/2}\exp\left(-\beta\Delta^{(\mathbf{n})}\right), \quad (43)$$

where $\Delta^{(n)}$ is the minimum value of Δ on the circle $\cos\chi = 0$ perpendicular to the direction **n**; $\sigma\Delta_0$ is the value of the derivative $\partial^2\Delta/\partial\varphi^2$ at the minimum point on the circle. From (39) and (43), we get

$$I_1/I_0 \approx (v^2/c^2\sqrt{\beta\Delta})\exp\{-\beta(\Delta^{(\mathbf{n})}-\Delta_0)\}. \quad (44)$$

Let us consider two limiting cases.

1. Let $I_1/I_0 \gg 1$ in (44). For the majority of "pure" superconductors, only this case is amenable to experimental investigation at the present time (since the attainable values of $\beta\Delta$ have the order of ~ 10). We note that the condition $(\beta\Delta)^{3/2} c/v_F \ll 1$ is presumed to be satisfied beforehand.

With the aid of (35) and (43) we obtain

$$\alpha_s \sim \omega_s(\mathbf{q})\sqrt{T/T_c}\exp\left(-\Delta^{(\mathbf{n})}/T\right). \quad (45)$$

Equation (45) shows that α_s increases exponentially with temperature, while the exponent depends on the direction. This conclusion is confirmed by experimental data given in references 5 and 6.

We proceed to the problem of establishing $\Delta(\mathbf{n})$ from the known function $\alpha_s(\mathbf{q})$. For simplicity, we consider the case of azimuthal symmetry of $\Delta(\mathbf{n})$. Let the direction corresponding to the minimum of $\Delta(\mathbf{n})$ coincide with the axis of symmetry, and $\Delta(\vartheta)$ increase monotonically from the pole to the equator. Then the following formula clearly holds:

$$\Delta(\vartheta)=\Delta(\pi/2-\vartheta). \quad (46)$$

Thus in this case the experimental measurements make it possible to establish $\Delta(\vartheta)$ completely and uniquely.

However, in the case in which some finite angle ϑ_0 corresponds to the minimum of $\Delta(\vartheta)$ on the sphere, the values of $\Delta(\vartheta)$ [and, consequently, $\ln\alpha_s(\vartheta)$] for the angles $\vartheta > \pi/2 - \vartheta_0$ will be identical and Eq. (46) makes it possible to establish $\Delta(\vartheta)$ only for the angles $\vartheta > \vartheta_0$ (the symmetry $\vartheta \rightleftarrows \pi - \vartheta$ is presumed everywhere). In particular, for $\vartheta_0 = \pi/2$, we can only establish the values of the minimum of Δ_0.

The situation considered above is obviously typical in the sense that in the establishment of $\Delta(\mathbf{n})$ according to $\alpha_s(\mathbf{q})$ "white spots" appear on the sphere. Detailed analysis of the different possibilities which arise in the absence of azimuthal symmetry, the presence of certain minima of $\Delta(\mathbf{n})$, etc., will be given in a separate communication.

2. Let $I_1/I_0 \ll 1$ in (44). Making use of (35) and (43), we get

$$\alpha_s(\mathbf{q})=A\nu_s(\mathbf{n})\,\omega_s(\mathbf{q})\,(c/v_F)^2 e^{-\beta\Delta_0}\sum|\cos^3\chi_0|^{-1}, \quad (47)$$

where χ is the angle between the direction q and the velocity at the minimum point of Δ, A is a constant of the order of unity (the summation is extended over all points of absolute minimum of Δ). In the derivation of Eq. (47), it was assumed that the coincidence of the values of $\Delta(\mathbf{n})$ and the different points of the minimum is not accidental but a consequence of the symmetry.

The behavior of α_S predicted by Eq. (47) is substantially different from the first case. We turn our attention to the anomalous behavior of α_S according to the directions corresponding to the small cos χ.*

It is interesting that for Nb, the transition temperature of which is about 8° K, the region for the second case is perhaps experimentally attainable at the present time and Eq. (47) can be checked experimentally.

Up to now we have assumed that the stereographic projection of the Fermi surface is an entire sphere. However, for open Fermi surfaces the stereographic projection is only part of a sphere. For surfaces of the "corrugated" type (see reference 14), the unoccupied part of the sphere can include large regions. If the sound is propagated in a direction perpendicular to the "unoccupied" region, then cos χ does not have small values on the Fermi surface and consequently the ultrasonic attenuation is described by Eq. (47).

We emphasize that in the given range of directions the ultrasonic attenuation is described by Eq. (47) under the condition $\beta\Delta \gg 1$, i.e., for temperatures much higher than required by the condition of the second case.

*Of course, cos χ may not be set equal to zero, since the condition $|\cos\chi| \geq c/v_F$ must be fulfilled.

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