DOUBLE DISPERSION RELATIONS FOR POTENTIAL SCATTERING

V. I. MAL'CHENKO

Dnepropetrovsk State University

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We study the analytic properties of the scattering matrix $T(k^2, t)$ as a function of t for real k^2, where t is the square of the momentum transfer, and k^2 is the square of the momentum. The potentials treated are of the form $F(r)r^{-1}e^{-\alpha r}$.

In this article we study the analytic properties in the complex t plane of the quantum mechanical amplitude $T(k^2, t)$ for elastic potential scattering, for real k^2. Here t is the square of the momentum transfer, and k^2 is the square of the momentum. In particular, for a potential of the form

$$V(r) = F(r)r^{-1}e^{-\alpha r}, \qquad \alpha > 0 \tag{1}$$

we are able to prove the following assertion: the scattering amplitude $T(k^2, t)$ can be extended analytically into the complex t plane for all real values of k^2, and the existence, location, and types of singularities are all determined by $F(r)$. Further, $T(k^2, t) \to 0$ as $t \to \infty$ in any direction other than along the positive real axis.

The restrictions on $F(r)$ depend both on our method of proof and on whether the scattering is nonrelativistic or relativistic (determined by the Klein-Gordon equation). In the nonrelativistic case it is sufficient to choose $F(r)$ such that the Fourier transform $\tilde{V}(k)$ of $V(r)$ exists, and such that if one writes

$$F(r) = \int_0^\infty e^{-rs} f(s)\,ds \tag{2}$$

then

$$\tilde{V}(k) = \int e^{i\mathbf{kr}} V(r)\,d\mathbf{r} = 4\pi \int_0^\infty \frac{f(s)\,ds}{(\alpha+s)^2 + k^2}. \tag{3}$$

In the relativistic case, in addition to (3) we require the existence of $\widetilde{V^2}(k)$, where

$$\widetilde{V^2}(k) = \int e^{i\mathbf{kr}} V^2(r)\,d\mathbf{r} = 4\pi \int_0^\infty \frac{f_1(s)\,ds}{(2\alpha+s)^2 + k^2}, \tag{4}$$

$$f_1(s) = \int_0^s dp \int_0^p d\tau\, f(\tau) f(p-\tau). \tag{5}$$

Henceforth the $f(s)$ and $f_1(s)$ functions of Eqs. (2) — (5) shall be restricted by the additional condition that $\lim rF(r) = 0$ in the nonrelativistic case and $\lim F(r) = 0$ in the relativistic case as $r \to 0$. We may remark, however, that the results we obtain would seem to be valid also for a wider class of functions. In the nonrelativistic case one may, in addition, include $F(r)$ functions whose transforms are the $\delta(s)$ function or its derivatives $\delta^{(n)}(s)$. In the relativistic case the $\delta^{(n)}(s)$ are admissible.

As an illustration of our method of proof we proceed with the case of relativistic scattering. The configuration-space representation of the scattering matrix $T_1(\mathbf{k}', \mathbf{k})$ has been obtained elsewhere.[1] The momentum space representation is

$$T_1(\mathbf{k}', \mathbf{k}) = T(k^2, t) = \widetilde{W}(|\mathbf{k}' - \mathbf{k}|) - \frac{1}{8\pi^3}\int \widetilde{W}(|\mathbf{k}' - \mathbf{k}_1|)\frac{d\mathbf{k}_1}{k_1^2 - k^2 - i\varepsilon}\widetilde{W}(|\mathbf{k}_1 - \mathbf{k}|)$$
$$+ \frac{1}{64\pi^6}\int \widetilde{W}(|\mathbf{k}' - \mathbf{k}_1|)\frac{d\mathbf{k}_1}{k_1^2 - k^2 - i\varepsilon} G_1(k; \mathbf{k}_1, \mathbf{k}_2) \times \frac{d\mathbf{k}_2}{k_2^2 - k^2 - i\varepsilon}\widetilde{W}(|\mathbf{k}_2 - \mathbf{k}|) - \frac{1}{64\pi^6}\int W(|\mathbf{k}' - \mathbf{k}_1|)$$
$$\times \frac{d\mathbf{k}_1}{k_1^2 - k^2 - i\varepsilon} G_2(k; \mathbf{k}_1, \mathbf{k}_2)\frac{d\mathbf{k}_2}{k_2^2 - k^2 - i\varepsilon}\widetilde{W}(|\mathbf{k}_2 - \mathbf{k}|). \tag{6}$$

Explicit expressions for G_1 and G_2 can be obtained by the Fredholm solution of the scattering problem.[2] For instance, G_1 is given by

$$G_1(k; \mathbf{k}_1, \mathbf{k}_2) = \frac{1}{\Delta(k)}\Big[\widetilde{W}(|\mathbf{k}_1 - \mathbf{k}_2|) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int \begin{vmatrix} \widetilde{W}(|\mathbf{k}_1 - \mathbf{k}_2|) & K_2(\mathbf{k}_1, \mathbf{x}_1) & \ldots & K_2(\mathbf{k}_1, \mathbf{x}_n) \\ K_2(\mathbf{x}_1, \mathbf{k}_2) & K_2(\mathbf{x}_1, \mathbf{x}_1) & \ldots & K_2(\mathbf{x}_1, \mathbf{x}_n) \\ \ldots & \ldots & \ldots & \ldots \\ K_2(\mathbf{x}_n, \mathbf{k}_2) & K_2(\mathbf{x}_n, \mathbf{x}_1) & \ldots & K_2(\mathbf{x}_n, \mathbf{x}_n) \end{vmatrix} d\mathbf{x}_1 \ldots d\mathbf{x}_n\Big], \tag{7}$$

where

$$\widetilde{W}(|\mathbf{k}_1 - \mathbf{k}_2|) = 2\sqrt{k^2 + m^2}\,\tilde{V}(|\mathbf{k}_1 - \mathbf{k}_2|) - \widetilde{V^2}(|\mathbf{k}_1 - \mathbf{k}_2|),$$

$$K_2(\mathbf{k}_1, \mathbf{x}_i) = \int d\mathbf{z}\, e^{-i\mathbf{k}_1\mathbf{z}} W(z)\frac{e^{ik|\mathbf{z}-\mathbf{x}_i|}}{4\pi|\mathbf{z}-\mathbf{x}_i|}W(x_i),$$

$$K_2(\mathbf{x}_i, \mathbf{k}_2) = \int d\mathbf{z}\frac{e^{ik|\mathbf{x}_i-\mathbf{z}|}}{4\pi|\mathbf{x}_i-\mathbf{z}|}W(z)e^{i\mathbf{k}_2\mathbf{z}},$$

and $\Delta(k)$ is a certain function of k. The G_2 function has the same properties as G_1, and we therefore refrain from writing it out.

We have now reduced our study of the analytic properties of $T(k^2, t)$ to the study of Eqs. (6) and (7). Proceeding with Klein,[3] we write

$$\mathbf{k}' = k(\cos(\theta/2), -\sin(\theta/2), 0),$$

$$\mathbf{k}_1 = k_1(\sin\gamma\sin\varphi, \cos\gamma\sin\varphi, \cos\varphi),$$

where θ is the scattering angle, and make use of (3) – (5). It can then be shown, for example, that for potentials of the form

$$e^{-\alpha r}, \quad re^{-\alpha r}, \ldots, r^n e^{-\alpha r} \tag{8}$$

$T(k^2, t)$ can be analytically extended, for all real k^2, into the t-plane cut along the real axis from $4\alpha^2$ to ∞ with poles of different orders at $t = \alpha^2$ and $t = 4\alpha^2$.

We now write down the usual dispersion relation for $T(E, t)$ in the absence of bound states (with $E = k^2 + m^2$):

$$T(E, t) = \frac{(E-m)(E-E_0)}{\pi} \mathrm{P} \times \int_m^\infty dE' \left[\frac{\operatorname{Im} T(E', t)}{(E'-m)(E'-E_0)(E'-E-i\varepsilon)} + \frac{\operatorname{Im} T_a(E', t)}{(E'+m)(E'+E_0)(E'+E)}\right] + \frac{m-E}{m-E_0}\operatorname{Re}T(E_0, t) + \frac{E-E_0}{m-E_0}T(m, t). \tag{9}$$

For potentials satisfying (8) this relation was obtained[1] for $-t < 4\alpha^2$. But both sides of (9) are analytic functions of t (for $E_0 > m$) for all t except the poles and the cut mentioned above. Hence (9) is valid for all t. Further, since the right side of (9) can be extended analytically into the complex E plane with cuts from $\pm m$ to $\pm\infty$, this equation establishes the analytic properties of $T(E, t)$ for complex E and complex t.

From these results we may now write the double dispersion relation

$$T(E, t) = \frac{(E-m)(E-E_0)}{\pi} \times \left[\mathrm{P}\int_m^\infty \frac{dE'}{(E'-E_0)(E'-m)(E'-E)} \int_{(2\varkappa)^2}^\infty \frac{\rho_1(E', t')}{t'-t} dt' + \int_m^\infty \frac{dE'}{(E'+E_0)(E'+m)(E'+E)} \int_{(2\varkappa)^2}^\infty \frac{\rho_2(E', t')}{t'-t} dt'\right] + \frac{m-E}{m-E_0}\operatorname{Re}T(E_0, t) + \frac{E-E_0}{m-E_0}T(m, t), \tag{10}$$

where $\rho_{1,2}(E', t')$ are real functions describing respectively, scattering of particles and antiparticles.

Several results can be obtained from Eq. (10). First, one can establish the analytic properties of the partial wave amplitudes by using the formula

$$A_l(E) = \frac{1}{2}\int_{-1}^{+1} P_l(\cos\theta)\, T(E, t)\, d\cos\theta,$$

$$t = -2k^2(1-\cos\theta).$$

Further, by comparing the analytic properties of $\operatorname{Im} T(E, t)$ with those that would have been obtained for the imaginary part of the meson-nucleon scattering amplitude if Lehmann's[4] procedure had been used, one finds (letting the nucleon mass approach infinity) that the radius of the meson-nucleon interaction is $\rho = \alpha^{-1} = 0.86 \times 10^{-13}$ cm. We have obtained this result elsewhere[5] in a somewhat different way.

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