THRESHOLD PHENOMENA IN SUPERCONDUCTORS

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Decay thresholds of elementary excitations are established for superconductors at absolute zero. It is shown that ultrasonic waves in a superconductor are not absorbed up to threshold frequency $\omega_t = 2\Delta$; at the threshold point the absorption jumps abruptly to a finite value $\sim 10^{-2}\,\omega_t$. Processes of sound emission and electron-hole pair production are considered. The relative role of these processes is clarified for various electron energies.

PHENOMENA of decay of elementary excitations in a superconductor have a number of peculiarities that distinguish them from similar phenomena in a normal metal. The presence of decay thresholds is especially characteristic for superconductors.

In the normal metal, phonons of suitably low frequencies can decompose into an electron-hole pair; an electron (hole) on the Fermi surface can produce a phonon, since the Fermi velocity v_F is much larger than the sound velocity c. Finally, the electron (hole) on the Fermi surface can eject a pair from the Fermi sphere

All these phenomena become possible in a superconductor only at certain threshold values of the energies of the elementary excitations. This is connected with the existence of a gap Δ in the energy spectrum of the electrons. Obviously, the decay of a phonon into an electron-hole pair is possible only for phonons with energies $\omega > 2\Delta$; the creation of an electron pair is possible for electrons with energies $\epsilon > 3\Delta$. Finally, the electron on the Fermi surface cannot excite a phonon, since the velocity of the electrons on the Fermi surface is equal to zero in a superconductor. Therefore, there exists an electron energy threshold for this phenomenon also. This threshold is very close to the origin of the spectrum (at a distance of the order $c^2\Delta/v_F^2$).

From the experimental viewpoint, the phonon decay threshold $\omega_t = 2\Delta$ is clearly of most interest, i.e., the threshold frequency of ultrasonic absorption ($\omega_t \sim 10^{10} - 10^{11}$ sec^{-1}), below which absorption vanishes at the temperature of absolute zero.

The purpose of the present research is the study of the decay phenomena mentioned, close to threshold.* The successive determination of the spectrum of the elementary excitations in the superconductor should be based on the solutions of the Dyson equations, to which we now proceed. We shall give a representation of these equations in a somewhat different form from that of Éliashberg,[1] which is, however, equivalent to it. Our representation appears to us to be more convenient.

1. THE DYSON EQUATIONS

As a starting point, we shall make use of Froelich's Hamiltonian in the form in which it was written in the research of Migdal,[2] and also the Green's function D for the phonon, introduced in the same paper [see Eqs. (1), (3) in reference 2]. The electrons in the superconductor are characterized by two functions:

$$G_{\alpha\beta}(\mathbf{p}, t-t') = i\langle T a_{\alpha\mathbf{p}}(t)\, a^{+}_{\beta\mathbf{p}}(t')\rangle, \tag{1}$$

$$F_{\alpha\beta}(\mathbf{p}, t-t') = \langle N+2\,|\,T a_{\alpha\mathbf{p}}(t)\, a_{\beta,-\mathbf{p}}(t')\,|\,N\rangle\, e^{2\mu t}. \tag{2}$$

(Here $\langle\ \rangle$ denotes averaging over the ground state of the system, $|N\rangle$ is the ground state of a system composed of N particles.) The function $F_{\alpha\beta}(\mathbf{p}, t-t')$ describes the formation of a condensed pair of two electrons or the decay of the pair into two holes.

We shall seek the functions G and F in the momentum representation. The set of Dyson

*In a recently published work,[1] Éliashberg considered the interaction of electrons and phonons in a superconductor. Threshold phenomena were not noted, since he identified the Green's function of the phonon with that obtained by Migdal[2] for the normal conductor, and relied on an inexact theorem on the adiabatic properties of the Green's functions for electrons.

equations for the functions G, F, D has the form

$$G(P) = G_0(P) + G_0(P)\Sigma_1(P)G(P) - G_0(P)\Sigma_2(P)F(P), \tag{3}$$

$$F(P) = G_0(-P+2\mu)\Sigma_1(-P+2\mu)F(P) + G_0(-P+2\mu)\Sigma_2(P)G(P), \tag{4}$$

$$D(Q) = D_2(Q) + D_0(Q)\Pi(Q)D(Q), \tag{5}$$

$$\Sigma_1(P) = \frac{1}{i(2\pi)^4}\int D(Q)[G(P-Q)\Gamma_1(P,Q) + F(P-Q)\Gamma_2(P,Q)]\,d^4Q \tag{6}$$

$$\Sigma_2(P) = \frac{1}{i(2\pi)^4}\int D(Q)[F(P-Q)\Gamma_1(P,Q) - G(P-Q)\Gamma_2(P,Q)]\,d^4Q, \tag{7}$$

$$\Pi(Q) = -\frac{1}{i(2\pi)^4}\int \{[G(P)G(P-Q) - F(P)F(P-Q)]\Gamma_1(P,Q) + G(P)[F(P-Q)+F(P+Q)]\Gamma_2(P,Q)\}\,d^4p. \tag{8}$$

The letters P and Q denote 4-vectors with components $P = (\mathbf{p}, \eta)$, $Q = (\mathbf{q}, \omega)$; $p = |\mathbf{p}|$, $q = |\mathbf{q}|$; μ denotes a 4-vector whose only non-zero component coincides with the chemical potential of the system, $\mu\,[\mu = (0, 0, 0, \mu)]$.

The system (3) — (8) is similar to that introduced by Belyaev[3] for a gas of interacting Bose particles below the condensation point. The functions $G_0(P)$, $D_0(Q)$ are given by the formulas

$$G_0(P) = \frac{1}{\xi_p - \eta - i\delta\theta(\xi_p)}, \quad \xi_p = \frac{p^2}{2} - \mu, \quad \eta = \varepsilon - \mu, \tag{9}$$

$$D_0(Q) = \frac{\lambda_0\pi^2}{p_0}\,\omega_q\left(\frac{1}{\omega_q - \omega - i\delta} + \frac{1}{\omega_q + \omega - i\delta}\right), \quad \omega_q = cq. \tag{10}$$

Equations cannot be written down for the vertex parts $\Gamma_1(P, Q)$, $\Gamma_2(P, Q)$. The procedure for their construction is completely analogous to the usual, but some pairs of vertices are joined not by the usual lines which represent the function G, but by lines with oppositely directed ends, which represent the function F. At each vertex, one electron line must enter, and the other must leave.

The quantities G, F, Σ_i, Γ_i are matrices over the spin indices. By virtue of the isotropy of space, $G_{\alpha\beta}$ has the form

$$G_{\alpha\beta}(p, \eta) = \delta_{\alpha\beta}G(p, \eta). \tag{11}$$

By the law of conservation of the projection of the spin, the diagonal elements of the matrix $\hat{F}$ are equal to zero. Also, from the isotropy of space, the other two elements are either equal or are opposite in sign. To decide on one of these possibilities, we note that it follows from (2) that

$$F_{\alpha\beta}(p, \eta) = -F_{\beta\alpha}(p, -\eta).$$

This equality yields $F_{\alpha\beta}(p, 0) = F_{\beta\alpha}(p, 0)$ for $\eta = 0$, whence we derive the result (from continuity) that the matrix $\hat{F}$ has the form:

$$\hat{F}(p, \eta) = \hat{\sigma}_2 F(p, \eta), \tag{12}$$

where $\hat{\sigma}_2$ is the Pauli matrix.

Equations (3) — (7) show that Σ_1 and Γ_1 have the same matrix structure as G, while Σ_2 and Γ_2 are similar to F.

2. THE WEAK-COUPLING APPROXIMATION

The theory of Bogolyubov[4] corresponds to the assumption $\lambda_0 \ll 1$. Setting $\Gamma_1 = 1$ and $\Gamma_2 = 0$ in the lowest approximation, and replacing D and G by D_0 and G_0 in Eqs. (6) and (7), we get the equations of weak coupling:

$$G(P) = \overline{G}_0(P) - \overline{G}_0(P)\Sigma_2(P)F(P), \quad F(P) = \overline{G}_0(-P+2\mu)\Sigma_2(P)G(P); \tag{13}$$

$$\overline{G}_0(P) = G_0(P)(1 + G_0(P)\Sigma_1(P))^{-1}. \tag{14}$$

It is not difficult to see that $\overline{G_0}(P)$ differs from $G_0(P)$ by the small renormalization of the velocity v_F on the Fermi surface and of the chemical potential μ. We shall assume these quantities to be renormalized and shall omit the bar in $\overline{G_0}$ in what follows.

The solution of the system (13) has the form

$$G(P) = \frac{\xi_p + \eta}{\varepsilon_p^2 - \eta^2 - i\delta} = \frac{u_p^2}{\varepsilon_p - \eta - i\delta} - \frac{v_p^2}{\varepsilon_p + \eta - i\delta}, \tag{15}$$

$$F(P) = \frac{\Sigma_2(P)}{(\varepsilon_p - \eta - i\delta)(\varepsilon_p + \eta - i\delta)}, \tag{16}$$

$$\varepsilon_p^2 = \xi_p^2 + \Sigma_2^2(P), \quad u_p^2 = \tfrac{1}{2}(1 + \xi_p/\varepsilon_p), \quad v_p^2 = \tfrac{1}{2}(1 - \xi_p/\varepsilon_p). \tag{17}$$

Equations (15) and (16) represent a generalization of the formulas of Gor'kov[5] to the case of electron-phonon interaction.

For $\Sigma_2(P)$ we have the equation

$$\Sigma_2(P) = \frac{2\lambda_0\pi^2}{i(2\pi)^4 p_0}\int \frac{\Sigma_2(P-Q)}{\varepsilon_{p-q}^2 - (\eta-\omega)^2 - i\delta}\,\frac{\omega_q^2}{\omega_q^2 - \omega^2 - i\delta}\,d^4Q. \tag{18}$$

In the integral of (18), as we shall establish below, the values of $|\mathbf{p} - \mathbf{q}|$ close to p_0 will play a role; more precisely, $|\mathbf{p} - \mathbf{q}| - p_0 \sim \Delta/p_0$, where $\Delta = \Sigma_2(p_0, \mu)$. We can therefore assume $\Sigma_2(p, \epsilon)$ to be independent of the spatial momentum p, and replace $\Sigma_2(p, \epsilon)$ by $C(\eta) = \Sigma_2(p_0, \mu + \eta)$.

FIG. 1

We transform to the variables $\epsilon_{p-q} = \epsilon'$, $\omega_q = \omega'$, and extend the integral over ϵ' to infinity. We get

$$C(\eta) = \frac{\lambda_0 \pi^2}{2i(2\pi)^3 \omega_0^2} \int_0^{\omega_0} \omega'^2 d\omega' \int_{-\infty}^{\infty} d\omega C(\eta - \omega) \left(\frac{1}{\omega' - \omega - i\delta} + \frac{1}{\omega' + \omega - i\delta} \right) \int_L \varepsilon' d\varepsilon' / \sqrt{\varepsilon'^2 - C^2(\eta - \omega)}\, (\varepsilon' - \eta + \omega - i\delta)(\varepsilon' + \eta - \omega - i\delta), \quad (19)$$

where the integration over ϵ' runs along the contour L enclosing two cuts from $\pm C(\eta - i\omega)$ to $\pm\infty$, as is shown in Fig. 1. The integral along the contour L can be replaced by the difference in the residues at the corresponding poles $\epsilon' = \pm(\eta - \omega) - i\delta$ if $(\eta - \omega)^2 > \Delta^2$, or by zero if $(\eta - \omega)^2 < \Delta^2$. We thus obtain

$$C(\eta) = \frac{\lambda_0}{8\omega_0^2} \int_0^{\omega_0} \omega'^2 d\omega' \int_{\Delta}^{\infty} \frac{C(\varepsilon')}{\sqrt{\varepsilon'^2 - C^2(\varepsilon')}} \left(\frac{1}{\omega + \varepsilon' + \eta} + \frac{1}{\omega - \varepsilon' + \eta} \right). \quad (20)$$

Here we have omitted $i\delta$ in the denominator, since it is not difficult to show that the imaginary part of Σ_2 is small in comparison with the real.

At small $\eta \lesssim \Delta$, Eq. (20) is identical with Bogolyubov's equation for $C(\eta)$ and qualitatively gives the same results for $\eta \gg \Delta$. The reasons for some divergence of Eq. (20) from Bogolyubov's equation were given by Éliashberg[1] and we shall not consider them here.

We now proceed to the calculation of successive approximations in the weak coupling theory. We shall be interested in decay processes of elementary excitations near the threshold. As Pitaevskiĭ[6] has pointed out, the damping, both in the case of weak and in the case of strong coupling, is determined by diagrams which correspond to the decay process under examination, wherein the distance to threshold appears as a really small parameter. The results of weak and strong coupling theories can be distinguished only when the spectrum close to threshold in the weak coupling theory splits up (in the analogous case in strong coupling theory, the threshold is the end point of the spectrum). Further results show that there is no splitting of the spectrum in a superconductor. Therefore, our results, which we obtained in the weak coupling case, also remain valid for strong coupling, if we assume $\lambda_0 \sim 1$ and replace the velocity v_F, the chemical potential μ and the value of the gap Δ by the renormalized values.

Moreover, these results will correctly describe the damping of quasi-particles far from the threshold if the electronic energy ξ is less than or of the order of the Debye energy ω_0.

This is related to the fact that the damping of phonons in the strong coupling case is determined by the process of decay into a pair, while the damping of the electrons in the given energy region is determined either by radiation of phonons or by pair creation.[2]

3. ULTRASONIC DAMPING

Kinematically, the decay of a phonon with wave vector q into an electron-hole pair is possible with satisfaction of the equality

$$\omega_q = \varepsilon_p + \varepsilon_{p-q}. \quad (21)$$

Eq. (21) can be satisfied only starting with $\omega_q = 2\Delta$, which also determines the decay threshold.

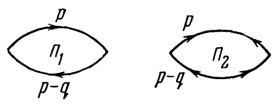

FIG. 2

For calculation of the damping associated with phonon-pair decay, we must take into account the contribution to the polarization operator of the phonon $\Pi(q, \omega)$ from the diagrams of Fig. 2.

$$\Pi(q, \omega) = \Pi_1 + \Pi_2, \quad (22)$$

$$\Pi_1(q, \omega) = \frac{i}{(2\pi)^4} \int d^4p \left[\frac{u_p^2}{\varepsilon_p - \eta - i\delta} - \frac{v_p^2}{\varepsilon_p + \eta - i\delta} \right] \times \left[\frac{u_{p-q}^2}{\varepsilon_{p-q} - \eta + \omega - i\delta} - \frac{v_{p-q}^2}{\varepsilon_{p-q} + \eta - \omega - i\delta} \right], \quad (23)$$

$$\Pi_2(q, \omega) = \frac{\Delta^2}{i(2\pi)^4} \int d^4p \,[(\varepsilon_p - \eta - i\delta)(\varepsilon_p + \eta - i\delta) \times (\varepsilon_{p-q} - \eta + \omega - i\delta)(\varepsilon_{p-q} + \eta - \omega - i\delta)]^{-1}. \quad (24)$$

Integrating over η and carrying out the substitution $\xi_p = \xi$, $\xi_{p-q} = \xi_1$, we get

$$\Pi_1(q, \omega) = \frac{1}{(2\pi)^2 q} \int_{-\infty}^{\infty} d\xi \int_{\xi - p_0 q}^{\xi + p_0 q} d\xi_1 \left(\frac{u_\xi^2 v_{\xi_1}^2}{\varepsilon_\xi + \varepsilon_{\xi_1} - \omega - i\delta} + \frac{v_\xi^2 u_{\xi_1}^2}{\varepsilon_\xi + \varepsilon_{\xi_1} + \omega - i\delta} \right), \quad (25)$$

$$\Pi_2(q, \omega) = \frac{\Delta^2}{4(2\pi)^2 q} \int_{-\infty}^{\infty} d\xi \int_{\xi - p_0 q}^{\xi + p_0 q} d\xi_1 \frac{1}{\varepsilon_\xi \varepsilon_{\xi_1}} \left(\frac{1}{\varepsilon_\xi + \varepsilon_{\xi_1} - \omega - i\delta} + \frac{1}{\varepsilon_\xi + \varepsilon_{\xi_1} + \omega - i\delta} \right). \quad (26)$$

In Eqs. (25) and (26), the integration over ξ_1 can be extended to $\pm\infty$ since $p_0 q \gg \omega_q$ in all cases, and just ξ, $\xi_1 \sim \omega_q$, appear in the essential region of integration in the calculation of $\mathrm{Im}\,\Pi(q, \omega)$. Taking this into account, we transform to the variables $\epsilon = \epsilon_\xi$, $\epsilon_1 = \epsilon_{\xi_1}$ and obtain*

$$\mathrm{Im}\,\Pi_1(q, \omega) = \frac{\pi}{(2\pi)^2 q} \int_{\Delta}^{\infty} d\varepsilon \int_{\Delta}^{\infty} d\varepsilon_1 \frac{\varepsilon \varepsilon_1 \delta(\varepsilon + \varepsilon_1 - |\omega|)}{\sqrt{(\varepsilon^2 - \Delta^2)(\varepsilon_1^2 - \Delta^2)}} \quad (27)$$

*We neglect the weak dependence of $C(\eta)$ on η.

$$\operatorname{Im}\Pi_2(q,\ \omega)=\frac{\pi}{(2\pi)^2 q}\int\limits_{\Delta}^{\infty}d\varepsilon\int\limits_{\Delta}^{\infty}d\varepsilon_1\frac{\Delta^2\delta(\varepsilon+\varepsilon_1-|\omega|)}{\sqrt{(\varepsilon^2-\Delta^2)(\varepsilon_1^2-\Delta^2)}}. \quad (28)$$

Then

$$\operatorname{Im}\Pi(q,\ \omega)=\frac{\pi}{(2\pi)^2 q}\int\limits_{\Delta}^{\infty}d\varepsilon\int\limits_{\Delta}^{\infty}d\varepsilon_1\frac{(\varepsilon\varepsilon_1+\Delta^2)}{\sqrt{(\varepsilon^2-\Delta^2)(\varepsilon_1^2-\Delta^2)}}\times\delta(\varepsilon+\varepsilon_1-|\omega|). \quad (29)$$

It is easy to establish from (29) that $\operatorname{Im}\Pi=0$ for $\omega<2\Delta$. We shall consider the behavior of $\operatorname{Im}\Pi(q,\omega)$ in two limiting cases. First, let $\omega=2\Delta(1+\alpha/2)$ (for $\alpha\ll 1$). We have

$$\operatorname{Im}\Pi=\frac{\pi}{(2\pi)^2 q}\int\limits_{\Delta}^{\Delta(1+\alpha)}\frac{d\varepsilon\,(\varepsilon\varepsilon_1+\Delta^2)}{\sqrt{(\varepsilon^2-\Delta^2)(\varepsilon_1^2-\Delta^2)}}, \quad (30)$$

where $\epsilon_1=|\omega|-\epsilon$. Making the substitution $\epsilon=\Delta(1+\eta)$ in (30), we obtain

$$\operatorname{Im}\Pi=\frac{\pi\Delta}{(2\pi)^2 q}\int\limits_{0}^{\alpha}\frac{d\eta}{\sqrt{\eta(\alpha-\eta)}}=\frac{\Delta}{4q}. \quad (31)$$

The ultrasonic damping is determined by the equation

$$D_0^{-1}-\Pi=0 \quad (32)$$

and has near threshold the form

$$\operatorname{Im}\omega_q=-\frac{\lambda_0\pi^2}{4}\frac{\Delta}{p_0 q}\omega_q=-\frac{\lambda_0\pi^2}{4}\frac{\Delta}{\sqrt{M}}. \quad (33)$$

Equation (33) shows that the damping at threshold increases abruptly from zero to a finite quantity. For $\omega_q=2\Delta$, the ratio $(\operatorname{Im}\omega_q)/\omega_q$ has a value $\sim 1/\sqrt{M}\sim 10^{-2}$.

In the discussion above, there was a certain lack of rigor. One can show that $\operatorname{Re}\Pi$ near the threshold $\omega=2\Delta$ has a logarithmic singularity of the form

$$(\Delta/4q)\ln[(2\Delta-\omega)/\omega_0],$$

and, strictly speaking, there is no discontinuity in the damping at threshold. However, the region in which the logarithmic term in $\operatorname{Re}\Pi$ [in the solution of Eq. (32)] is appreciable is very small:

$$|\omega-\omega_0|\sim\omega_0\exp(-A/\lambda_0\sqrt{M}),$$

where A is a quantity of the order of unity. Clearly, the physical smearing of the jump in the damping in such a region is completely imperceptible.

In the other limiting case $\omega\gg 2\Delta$, we can set $\Delta=0$ in (30), from which we easily obtain

$$\operatorname{Im}\Pi(q,\ \omega)=|\omega|/4\pi q. \quad (34)$$

This result is identical with the result of Migdal [see Eq. (12) of reference 2].

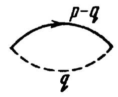

FIG. 3

4. PHONON EMISSION BY AN ELECTRON

The conservation laws in this process yield

$$\varepsilon_p=\varepsilon_{\mathbf{p}-\mathbf{q}}+\omega_q. \quad (35)$$

In a superconductor, the velocity of the electron is zero on the Fermi surface. Therefore emission of a phonon is possible only if we start with the value of the momentum $p_1=p_0+(\Delta/p_0)(c/v_F)$ for which the velocity of the electron becomes equal to the velocity of sound. In this case, Eq. (35) can for the first time be satisfied.

We find the damping of the electron excitations corresponding to this process by calculating the contribution to the "self energy" of the electron $\Sigma_1(p,\eta)$ (Fig. 3):

$$\Sigma_1(p,\ \eta)=\frac{1}{i(2\pi)^4}\int G_0(P-Q)D_0(Q)\,d^4Q. \quad (36)$$

Calculation of the integral (36) leads to the result:

$$\operatorname{Im}\Sigma_1(p,\ \eta)=\frac{\lambda_0\pi}{4pp_0}\int\limits_{0}^{q_m}q\omega_q dq\int\limits_{\xi_{|\mathbf{p}-\mathbf{q}|}}^{\xi_{\mathbf{p}+\mathbf{q}}}\delta(\varepsilon_{\xi_1}+\omega_q-|\eta|)\,d\xi_1. \quad (37)$$

If we substitute $\eta=\epsilon_p$ in (37), it is then evident that the argument of the δ function can vanish only if $d\epsilon/dp\geq d\omega/dq=c$. The value of $\eta=\eta_0\approx\Delta(1+c^2/2v_F^2)$ is the threshold of the process. It is not difficult to transform (3) to the form

$$\operatorname{Im}\Sigma_1(p,\ \eta)=\frac{\lambda_0\pi}{4pp_0}\sigma(\eta-\eta_0)\int\limits_{\eta_0}^{\eta}\frac{\varepsilon(\eta-\varepsilon)^2}{(\varepsilon^2-\Delta^2)^{1/2}}d\varepsilon=\frac{\lambda_0\pi}{4pp_0}\sigma(\eta-\eta_0)\int\limits_{0}^{q_\eta}\frac{q\omega_q\varepsilon_q dq}{(\varepsilon_q^2-\Delta^2)^{1/2}}. \quad (38)$$

We consider the behavior of $\operatorname{Im}\Sigma_1(p,\eta)$ in three cases.

1) $\eta=\eta_0(1+\alpha)$, $\alpha\ll c^2/v_F^2\sim 1/M$. Calculation of the integral (38) gives

$$\operatorname{Im}\Sigma_1(p_0,\ \eta)=\frac{\lambda_0\pi}{12\sqrt{2}}\left(\frac{\Delta}{\omega_0}\right)^2\Delta\alpha^3. \quad (39)$$

2) $\eta=\eta_0(1+\alpha)$, $1/M\ll\alpha\ll 1$,

$$\operatorname{Im}\Sigma_1(p_0,\ \eta)=\frac{\lambda_0\pi}{10\sqrt{2}}\left(\frac{\Delta}{\omega_0}\right)^2\Delta\alpha^{5/2}. \quad (40)$$

The latter result is identical, with accuracy up to a numerical factor of the order of unity, with the result of Éliashberg [see reference 1, Eq. (30)].

3) For $\eta\gg\Delta$, Eq. (38) goes over into the corresponding equation for the normal metal.

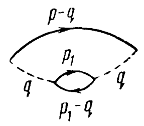

FIG. 4

5. CREATION OF AN ELECTRON-HOLE PAIR BY AN ELECTRON

The threshold of this process is obviously $\eta = 3\Delta$. By the conservation law applicable to the given case,

$$\varepsilon_p = \varepsilon_{\mathbf{p}-\mathbf{q}} + \varepsilon_{p_1} + \varepsilon_{\mathbf{p}_1-\mathbf{q}}. \quad (41)$$

The contribution to $\Sigma_1(p, \eta)$ corresponding to this process is given by the diagram of Fig. 4:

$$\Sigma_1(p, \eta) = \frac{\lambda_0^2}{4ip_0^2}\int d^4Q\left(\frac{\omega_q^2}{\omega_q^2-\omega^2-i\delta}\right)^2 \Pi(q, \omega)$$
$$\times\left(\frac{u_{\mathbf{p}-\mathbf{q}}^2}{\varepsilon_{\mathbf{p}-\mathbf{q}}-\eta+\omega-i\delta}-\frac{v_{\mathbf{p}-\mathbf{q}}^2}{\varepsilon_{\mathbf{p}-\mathbf{q}}+\eta-\omega-i\delta}\right), \quad (42)$$

where $\Pi(q, \omega)$ is determined by Eqs. (22), (23), and (24). It is clear from (41) that in pair creation all four energies should be of the same order of magnitude. Therefore, the values $\omega \sim \eta$ are important in the integral in (42). If $\eta \ll \omega_0$, then $\omega \ll \omega_q$. By virtue of (42), for $\eta \ll \omega_0$, we can transform to

$$\Sigma_1(p, \eta) = \frac{\lambda_0^2}{4ip_0^2}\int d^4Q\,\Pi(q, \omega)\left(\frac{u_{\mathbf{p}-\mathbf{q}}^2}{\varepsilon_{\mathbf{p}-\mathbf{q}}-\eta+\omega-i\delta}\right.$$
$$\left.-\frac{v_{\mathbf{p}-\mathbf{q}}^2}{\varepsilon_{\mathbf{p}-\mathbf{q}}+\eta-\omega-i\delta}\right). \quad (43)$$

We change to the variables $\epsilon = \epsilon_{p-q}$ and extend the integration over ϵ to the interval (Δ, ∞). The integration over ϵ is completed without difficulty, and yields

$$\int_\Delta^\infty\left(\frac{1}{\varepsilon-\tilde{\eta}-i\delta}-\frac{1}{\varepsilon+\tilde{\eta}-i\delta}\right)\frac{\varepsilon d\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}$$
$$= i\pi\int_\Delta^\infty[\delta(\varepsilon-\tilde{\eta})-\delta(\varepsilon+\tilde{\eta})]\frac{\varepsilon\, d\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}, \quad (44)$$

where $\tilde{\eta} = \eta - \omega$. We have carried out the integration over ϵ to the end here, in order to give a smoother form to the result. Substituting (44) in (43), we get

$$\operatorname{Im}\Sigma_1(p, \eta) = \frac{\pi\lambda_0^2}{2pp_0^2}\int_{-\infty}^{\infty} d\omega \int_0^{q_m} qdq\operatorname{Im}\Pi(q, \omega)$$
$$\times\int_\Delta^\infty[\delta(\eta-\omega-\varepsilon)-\delta(\eta+\omega+\varepsilon)]\frac{\varepsilon d\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}. \quad (45)$$

Here we have made use of the parity of $\Pi(q, \omega)$ relative to ω.

Now let $\eta > 0$. Making use of Eq. (29) for Im Π, we get from (45)

$$\operatorname{Im}\Sigma_1(p, \eta) = \frac{\lambda_0^2}{2}\frac{q_m}{4\pi pp_0^2}\int_\Delta^\infty d\varepsilon_1\int_\Delta^\infty d\varepsilon_2$$
$$\times\int_\Delta^\infty d\varepsilon_3\frac{(\varepsilon_1\varepsilon_2+\Delta^2)\,\varepsilon_3}{[(\varepsilon_1^2-\Delta^2)(\varepsilon_2^2-\Delta^2)(\varepsilon_3^2-\Delta^2)]^{1/2}}\delta(\eta-\varepsilon_1-\varepsilon_2-\varepsilon_3). \quad (46)$$

Near threshold, we have $\eta = 3\Delta(1+\alpha/3)$, $\epsilon_i = \Delta(1+\eta_i)$, $(\alpha, \eta_i \ll 1)$, and Eq. (46) gives

$$\operatorname{Im}\Sigma_1(p_0, \eta) = \frac{\lambda_0}{16\pi\sqrt{2}}\frac{q_m\Delta^2}{p_0^3}\int_0^\alpha d\eta_1$$
$$\times\int_0^{\alpha-\eta_1}\frac{(\eta_1+\eta_2)\,d\eta_2}{[\eta_1\eta_2(\alpha-\eta_1-\eta_2)]^{1/2}} = \frac{\lambda_0^2}{12\sqrt{2}}\frac{q_m}{p_0}\frac{\Delta}{p_0^2}\Delta\alpha^{3/2}. \quad (47)$$

For $\alpha \sim 1$, Im $\Sigma_1 \sim \lambda_0^2\Delta^2/p_0^2$. Comparing (47) with (40), we get the ratio of the frequency of emission of the phonons ν_1 to the frequency of pair creation ν_2: $\nu_1/\nu_2 = (\Delta/\omega_0)(p^2/\omega_0)$ (we assume $\lambda_0 \sim 1$ here). Under real conditions, $\Delta/\omega_0 \sim \omega_0/p_0^2 \sim 10^{-2}$. Therefore, the contribution to the damping made by the pair creation process increases rapidly beyond threshold to a value exceeding the damping due to phonon emission.

If $\eta \gg \Delta$, we must set $\Delta = 0$ in the integral (46), whence

$$\operatorname{Im}\Sigma_1(p_0, \eta) \sim (\lambda_0^2/M)\,\omega_0(\eta/\omega_0)^2. \quad (48)$$

This value becomes (for $\lambda_0 \sim 1$) of the same order as the damping due to phonon emission ($\operatorname{Im}\Sigma_1 \sim (\lambda_0/\sqrt{M})\,\omega_0(\eta/\omega_0)^3$) for $\eta/\omega_0 \sim M^{-1/2}$. Equation (48) is identical with Eq. (31) of Migdal's research, with accuracy up to a numerical factor.[2]

Thus, for energies η lying in the range $(\Delta, 3\Delta)$, the electrons radiate sound. In the interval $(3\Delta, \omega_0^2/p_0^2)$ the decay of the electron is principally in the form of pair creation. Finally, in the range $(\omega_0^2/p_0^2, \omega_0)$, phonon emission plays the principal role.

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