SOME PROPERTIES OF DRESSED PARTICLE OPERATORS IN FIELD THEORY

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We investigate the properties of dressed particle operators, defined as producing single particle states when acting on the vacuum. Haag[1] has shown that the dressed particle creation operators have a strong-convergence limit for infinite time. We show that this is also true for the annihilation operators. Assuming the possibility of a coupling-constant expansion, we show that the dressed operators cannot satisfy the causality condition in its usual form.

1. INTRODUCTION

If one could construct a field theory based on "dressed" particle operators (as opposed to the usual "bare" particle operators), one would no longer need to go through the renormalization procedure; in principle such an approach, making use of no nonphysical quantities, would be the more systematic one. Nevertheless, the actual use of dressed operators is still restricted to the Schrödinger representation, for the properties of the corresponding Heisenberg operators are not yet well enough understood. It is known only that for a large time the dressed creation operators converge strongly to the so-called in and out creation operators[1] and that the dressed operators cannot satisfy the bare-operator commutation relations so long as the interaction fails to vanish.[1,2]

In this communication we establish some other properties of dressed operators. Section 2 discusses some equivalent definitions of dressed operators. Section 3 extends Haag's results[1] to annihilation operators [strictly speaking, we shall prove not that the dressed annihilation operators converge strongly to the bare one, but, like Haag, the somewhat weaker result of Eq. (13)]. Section 4 is devoted to causality properties of dressed operators. In it we show that these operators cannot satisfy the causality condition in its usual form, i.e., that their commutator does not vanish on a space-like surface. The proof is based on perturbation theory.

For simplicity we shall deal only with scalar particles of a single kind.

2. THE CONDITION THAT THE OPERATORS DESCRIBE DRESSED PARTICLES

Let us define a local Lorentz-invariant dressed particle operator $Q(x)$ by the condition that

$$Q^{(+)}(k,\ t)\,|0\rangle = |k\rangle. \tag{1}$$

Here $|0\rangle$ is the physical vacuum, $|k\rangle$ is a single-particle state corresponding to 4-momentum k (with $k^2 = \mu^2$) and such that

$$\langle k\,|\,k'\rangle = 2k_0\delta(\mathbf{k}-\mathbf{k}'), \tag{2}$$

and, as usual,

$$Q^{(\pm)}(k,\ x_0) = \mp i\int d^3x Q(x)\frac{\overleftrightarrow{\partial}}{\partial x_0}e^{\mp ikx} \equiv \mp i\int d^3x\left\{Q(x)\frac{\partial}{\partial x_0}e^{\mp ikx} - e^{\mp ikx}\frac{\partial}{\partial x_0}Q(x)\right\}. \tag{3}$$

Condition (1) agrees well with the physical meaning of a dressed operator.

We now define in and out operators according to

$$A_{in}^{out}(x) = Q(x) + \int \Delta_{ret}^{adv}(x-x')\,j(x')\,d^4x',\ \ j(x') \equiv (\Box - \mu^2)\,Q(x'). \tag{4}$$

It is known[1-3] that the $A_{in}^{out}(k,\ t)$ operators are independent of t and that they satisfy the usual free-particle commutation relations. It is known also that

$$A_{in}^{(+)}(k_1)\ \ldots\ A_{in}^{(+)}(k_n)\,|0\rangle \equiv |k_1\ \ldots\ k_n\rangle_{in}$$

is an energy-momentum eigenstate describing n particles with 4-momenta $k_1 \ldots k_n$ (with $k_i^2 = \mu^2$) before collision (and similarly for out operators). Further,

$$A_{in}^{(+)}(k)\,|0\rangle = A_{out}^{(+)}(k)\,|0\rangle = |k\rangle.$$

The relation between $Q^{(\pm)}(k,\ t)$ and, for instance, $A_{in}^{(\pm)}(k)$ is, according to (3) and (4), of the form

$$A_{in}^{(\pm)}(k) = Q^{(\pm)}(k,\ t) \pm i\int d^4x\,\theta(t-x)\,j(x)\,e^{\mp ikx}. \tag{5}$$

Condition (1) that the operators describe dressed particles states that

$$[A_{in}^{(+)}(k) - Q^{(+)}(k, t)]|0\rangle = i\int d^4x\theta(t-x)\,j(x)e^{-ikx}|0\rangle = 0,$$

from which it follows immediately that

$$j(x)|0\rangle = 0. \quad (6)$$

Note that this equation has been obtained as a consequence of (1). It is clear also that (1) is a consequence of (6). Furthermore, it is seen from (5) that (6) implies also that

$$[A_{in}^{(-)}(k) - Q^{(-)}(k, t)]|0\rangle = 0,$$

so that

$$Q^{(-)}(k, t)|0\rangle = 0. \quad (7)$$

Thus conditions (1), (6), and (7) are shown to be equivalent. Any one of them may be used as the condition that the operators describe dressed particles.

We remark also that together (1) and (7) lead to yet another form of this condition, likewise equivalent to those above, namely

$$Q(x)|0\rangle = A_{in}^{out}(x)|0\rangle. \quad (8)$$

This condition was used by Greenberg.[2]

As a further remark we point out that all this can also be formulated in terms of state vectors.[4,5] To do this we introduce the asymptotic states Ψ^a, satisfying the conditions

$$\langle\Psi_1^a|\Psi_2^a\rangle = G(1, 2) = \delta(1, 2) + g(1, 2),$$

$$\langle\Psi_1^a|2\rangle_{in} = \delta(1, 2) + T^{(\pm)}(1, 2)/(E_1 - E_2 \mp i0), \quad (9)$$

(where $g(1, 2)$ and $T^{(\pm)}(1, 2)$ are smooth functions) as well as the requirement that the vacuum and single-particle states of Ψ^a are $|0\rangle$ and $|k\rangle$, respectively. The relation of this to the previously defined dressed operators is given by

$$\Psi^{ac}(k_1 \ldots k_n) = Q^{(+)}(k_1, 0) \ldots Q^{(+)}(k_n, 0)|0\rangle. \quad (10)$$

3. STRONG CONVERGENCE

To study the behavior of the dressed operators as $t \to \pm\infty$, we go over in the usual way to wave packets and introduce the $Q_\alpha^{(\pm)}(x_0)$ operators, which are defined analogously to those of Eq. (3), except that $e^{\mp ikx}$ is replaced by $f_\alpha^{(\pm)}(x)$, a positive- or negative-frequency solution of the Klein-Gordon equation with mass μ, orthonormalized according to

$$\pm i\int d^3x f_\alpha^{(\pm)*}(x)\,\frac{\overleftrightarrow{\partial}}{\partial x_0}\,f_\beta^{(\pm)}(x) = \delta_{\alpha\beta} \quad (11)$$

and such that $(f_\alpha^{(+)})^* = f_\alpha^{(-)}$.

By postulating the behavior of the vacuum expectation values $Q(x)$ for equal times, Haag[1] has shown that

$$Q_{\alpha_1}^{(+)}(t) \ldots Q_{\alpha_n}^{(+)}(t)|0\rangle \rightrightarrows |\alpha_1 \ldots \alpha_n\rangle_{in\,(out)}. \quad (12)$$

Here the symbol $\rightrightarrows$ indicates strong convergence, and $|\alpha_1 \ldots \alpha_n\rangle_{in(out)}$ is an n-particle in (or out) state in which the wave function of the m-th particle in x-space is $f_{\alpha_m}(x)$.

We shall show that if the vacuum expectation values behave according to Haag's postulate, then in addition to (12) we have

$$Q_{\beta_1}^{(-)}(t) \ldots Q_{\beta_n}^{(-)}(t)\,Q_{\alpha_1}^{(+)}(t) \ldots Q_{\alpha_m}^{(+)}(t)|0\rangle$$

$$\underset{t\to-(+)\infty}{\rightrightarrows} \begin{cases} 0, & m < n \\ \sum \delta_{\beta_1\alpha_{i_1}} \ldots \delta_{\beta_n\alpha_{i_n}}|\alpha_{i_{n+1}} \ldots \alpha_{i_m}\rangle_{in\,(out)}, & m \geqslant n \end{cases} \quad (13)$$

(the sum is taken over all permutations $i_1 \ldots i_n$ of $1 \ldots m$).

It is known that (12) does not, strictly speaking, imply

$$Q_\alpha^{(+)} \rightrightarrows A_{\alpha\, in\,(out)}^{(+)} \quad \text{as} \quad t \to -(+)\infty,$$

although this convergence seems highly probable (see, for instance, Haag[1]). Similar considerations relating to (13) lead one to believe that

$$Q_\alpha^{(-)} \rightrightarrows A_{\alpha\, in\,(out)}^{(-)} \quad \text{as} \quad t \to -(+)\infty$$

and that therefore the difference between $Q(x)$ and $A_{in\,(out)}(x)$ converges strongly to zero as $t \to -(+)\infty$.

Equation (13) is proved in analogy with Haag's proof of (12). For simplicity we consider the case $m = n = 1$. Consider the expression

$$\langle 0|[Q_\beta^{(-)}(t)Q_\alpha^{(+)}(t) - Q_\alpha^{(+)}(t)Q_\beta^{(-)}(t) - \delta_{\alpha\beta}]^+ \times [Q_\beta^{(-)}(t)Q_\alpha^{(+)}(t) - Q_\alpha^{(+)}(t)Q_\beta^{(-)}(t) - \delta_{\alpha\beta}]|0\rangle \quad (14)$$

for large $|t|$. As Haag has shown, the vacuum expectation value of products of $Q_\alpha^{(\pm)}(t)$ operators reduce, as $|t| \to \infty$ (up to terms of order t^{-3}) to the sum of products of vacuum expectation values of pairs of $Q_\alpha^{(+)}(t)$ and $Q_\beta^{(-)}(t)$ operators taken in the same order as in the original product. In particular, for (14) we have, as $|t| \to \infty$,

$$\begin{aligned} &\langle 0|Q_\alpha^{(-)}Q_\beta^{(+)}|0\rangle\langle 0|Q_\beta^{(-)}Q_\alpha^{(+)}|0\rangle \\ &- \langle 0|Q_\beta^{(+)}Q_\alpha^{(-)}|0\rangle\langle 0|Q_\beta^{(-)}Q_\alpha^{(+)}|0\rangle \\ &- \langle 0|Q_\alpha^{(-)}Q_\beta^{(+)}|0\rangle\langle 0|Q_\alpha^{(+)}Q_\beta^{(-)}|0\rangle \\ &+ \langle 0|Q_\beta^{(+)}Q_\alpha^{(-)}|0\rangle\langle 0|Q_\alpha^{(+)}Q_\beta^{(-)}|0\rangle \\ &- 2\delta_{\alpha\beta}\langle 0|Q_\alpha^{(-)}Q_\beta^{(+)}|0\rangle + 2\delta_{\alpha\beta}\langle 0|Q_\alpha^{(+)}Q_\beta^{(-)}|0\rangle + \delta_{\alpha\beta}. \end{aligned}$$

We now find by using conditions (1) and (17) that this expression vanishes, or that

$$\lim_{|t|\to\infty} \| [Q_\beta^{(-)}(t) Q_\alpha^{(+)}(t) - Q_\alpha^{(+)}(t) Q_\beta^{(-)}(t) - \delta_{\alpha\beta}] |0\rangle \| = 0.$$

On the other hand, we know that $Q_\beta^{(-)}(t)|0\rangle = 0$, so that

$$Q_\beta^{(-)}(t) Q_\alpha^{(+)}(t)|0\rangle \underset{t\to\pm\infty}{\rightrightarrows} \delta_{\alpha\beta}|0\rangle,$$

q.e.d.

4. CAUSAL PROPERTIES OF DRESSED OPERATORS

It is well known that the condition that $Q(x)$ be a dressed operator does not define it uniquely. One is then led naturally to look for additional restrictions which will pick out the most "physical" or "convenient" of all possible $Q(x)$ operators. One would suppose that the causality condition would serve this function (see, for instance, Greenberg[2]); we write the condition in the form

$$[Q(x), Q(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0. \tag{15}$$

In this section we will show that dressed operators cannot satisfy (15). Since the commutation relations of the $Q(x)$ operators are not known, we shall make use of perturbation theory. Actually what we will show is that the condition that the operators describe dressed particles leads to

$$[A_{in(out)}(y), j(x)] \neq 0 \quad \text{for} \quad (x-y)^2 < 0. \tag{16}$$

This means that in the first approximation in the coupling constant g, the commutator $[Q(y), j(x)] \neq 0$ for $(x-y)^2 < 0$. It follows then that $[Q(y), Q(x)] \neq 0$ for $(x-y)^2 < 0$, also in the first approximation in g. If the perturbation theory expansion converges at least asymptotically, this means that the causality condition (15) cannot be fulfilled (except perhaps for special values of g such that the higher approximations cancel the first).

We proceed to prove (16). Let us expand the current $j(x)$ in a Fock series in the A_{in} (or, equivalently, A_{out}) operators:

$$j(x) = \sum_{m,n=0}^{\infty} \int \frac{d^3p_1}{2p_{10}} \cdots \frac{d^3q_1}{2q_{10}} \cdots \frac{f_{mn}(p_1\ldots p_m|q_1\ldots q_n|x)}{\sqrt{m!\,n!}}$$
$$\times A_{in}^{(+)}(p_1)\ldots A_{in}^{(+)}(p_m) A_{in}^{(-)}(q_1)\ldots A_{in}^{(-)}(q_n) \tag{17}$$

(here $p_i^2 = q_k^2 = \mu^2$). Considerations of Lorentz invariance lead to

$$f_{mn}(p_1\ldots p_m|q_1\ldots q_n|x) = \exp\left\{i\left(\sum p - \sum q\right)x\right\}\varphi_{mn}(p_1\ldots p_m|q_1\ldots q_n),$$

where φ_{mn} is a scalar function symmetric in the p_i and q_i separately. Because the current is Hermetian,

$$\varphi_{mn}(p_1\ldots p_m|q_1\ldots q_n) = \varphi_{nm}^*(q_1\ldots q_n|p_1\ldots p_m).$$

Let us think of the φ_{mn} functions as expanded in a power series in the coupling constant, and let us consider only those terms linear in g (which we shall denote henceforth simply as φ_{mn}). We remark that all the φ_{mn} cannot vanish in the first approximation in g, for this would contradict the very meaning of a coupling constant.

As a basis for proving (16) we shall use conditions (6), which leads obviously to

$$\varphi_{m0} = \varphi_{0m} = 0. \tag{18}$$

The commutation relations of the A_{in} are well known, and there is no difficulty in calculating the matrix elements of the commutator of $A_{in}(y)$ and $j(x)$ between any in states. We obtain

$$Z_{mn}(y, x) = \frac{1}{\sqrt{m!\,n!}}\,{}_{in}\langle p_1\ldots p_m|[A_{in}(y), j(x)]|q_1\ldots q_n\rangle_{in}$$
$$= \frac{1}{(2\pi)^{3/2}}\exp\left\{i\left(\sum p - \sum q\right)x\right\}\int\frac{d^3k}{2k_0}$$
$$\times \{\varphi_{m+1,n}(kp_1\ldots p_m|q_1\ldots q_n)e^{-ik(y-x)}\sqrt{m+1}$$
$$- \varphi_{m,n+1}(p_1\ldots p_m|kq_1\ldots q_n)e^{ik(y-x)}\sqrt{n+1}\} \tag{19}$$

(in the integrand $k^2 = \mu^2$).

Let $m = 0$, $n \neq 0$. It then follows from (18) that

$$Z_{0n}(y, x) = (2\pi)^{-3/2}\exp\left\{-ix\sum q\right\}\int\frac{d^3k}{2k_0}e^{-ik(y-x)}\varphi_{1n}(k|q_1\ldots q_n). \tag{20}$$

The right side of (20) cannot vanish for $(x-y)^2 < 0$ so long as $\varphi_{1n} \not\equiv 0$. Indeed, for $x_0 = y_0$ we are left with the ordinary Fourier transform. If the right side is to have a δ-function singularity, the integrand must contain a polynomial in $|k|$, which is clearly impossible because of the $2k_0$ in the denominator. Thus if $[A_{in}(y), j(x)] = 0$ for $(x-y)^2 < 0$, we obtain

$$\varphi_{1n} = \varphi_{n1} = 0.$$

Now consider the case $m = 1$, $n \neq 0, 1$. We have

$$Z_{1n} = \sqrt{2}(2\pi)^{-3/2}\exp\left\{i\left(p - \sum q\right)x\right\}$$
$$\times\int\frac{d^3k}{2k_0}e^{-ik(y-x)}\varphi_{2n}(k, p|q_1\ldots q_n).$$

Similar considerations lead to the conclusion that

$$\varphi_{2n} = \varphi_{n2} = 0.$$

Continuing in this way we can prove finally that if $[A_{in}(y), j(x)] = 0$ for $(x-y)^2 < 0$, all the φ_{mn} functions vanish, contrary to assumption. Thus we have proven (16).

In conclusion we should like to mention that it seems to us quite natural that the dressed operators do not satisfy the causality condition in the form of (15). This must, of course, be related to the smearing out of the physical particles over some region whose radius is of the order of $1/\mu$. This behavior of the dressed operators, however, is a serious obstacle to using them in constructing a Lorentz invariant formalism for scattering theory. To obtain such a formalism is the foremost problem in the theory of dressed particles.

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[3] W. Zimmerman, Nuovo cimento **10**, 597 (1958).

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[5] Yu. V. Novozhilov, JETP **35**, 742 (1958), Soviet Phys. JETP **8**, 515 (1959); Vestnik, Leningrad State Univ. **22**, 93 (1958); Nucl. Phys. **15**, 469 (1960).

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