HYDRODYNAMIC STABILITY OF A LIQUID LAYER ON A VERTICAL WALL

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We consider the flow of a liquid layer on a vertical wall under the influence of gravitation. The stability of the plane flow preceding wave flow is investigated in the first approximation.

REYNOLDS' number, given by the equation

$$\mathrm{Re} = 4Q/\gamma, \tag{1}$$

determines the way in which a thin layer of liquid will flow down a vertical wall under the influence of gravity. In Eq. (1), Q is the volume of liquid crossing unit width perpendicular to the flow per second and γ is the kinematic viscosity. The flow is two dimensional. Up to Reynolds numbers of about 1500, the flow is laminar and such that the relation

$$Q = ga^3/3\gamma \tag{2}$$

holds, where a is the thickness of the layer and g is the acceleration due to gravity. Equation (2) is derived for flow parallel to the wall and has been verified experimentally, provided that a is taken to be the mean thickness of the layer. However, the instantaneous thickness can be greater. Several investigators[1] have confirmed the hypothesis that there are waves on the surface of such a layer. Waves can exist for Reynolds numbers ranging from 25 to 1500.

P. L. Kapitza has done both theoretical[1] and experimental[2] work on the wave flow. He used an original method to find the least value of Re for which the wave flow was stable. His result agrees well with experimental data.

The purpose of the present paper is to investigate the stability of the parallel flow preceding the wave flow. This problem is different from that of P. L. Kapitza: the wave flow can appear at a Reynolds number different from that at which it disappears.

THE METHOD OF SMALL PERTURBATIONS

We proceed in the usual way by introducing small disturbances in the flow. Let x, y be coordinates parallel to the wall (and along the flow) and perpendicular to the wall (away from it) respectively; let u and v be the corresponding velocities. In the undisturbed flow the speed is given by

$$U = \frac{g}{\gamma}\left(ay - \frac{y^2}{2}\right). \tag{3}$$

Now let this flow be perturbed:

$$u = U + w, \qquad v = 0 + v, \qquad p = P + \pi. \tag{4}$$

The perturbed flow is described by the equation of continuity and the linearized Navier-Stokes equations. These are as follows, differentiation with respect to y being denoted by a prime:

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + vU' = -\frac{1}{\rho}\frac{\partial \pi}{\partial x} + \gamma\left(\frac{\partial^2 w}{\partial x^2} + w''\right),$$
$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{1}{\rho}\pi' + \gamma\left(\frac{\partial^2 v}{\partial x^2} + v''\right), \qquad \frac{\partial w}{\partial x} + v' = 0. \tag{5}$$

For periodic perturbations,

$$w, v \sim \exp[in(x - kt)] \tag{6}$$

the system of equations (5) becomes the Orr-Sommerfeld equation.[3]

Let us formulate the boundary conditions. At the wall,

$$y = 0, \qquad v = 0, \qquad v' = 0. \tag{7}$$

Let us assume that at the surface there is no interaction with the gas and the tangential stress is zero:

$$w' + \partial v/\partial x = 0. \tag{8}$$

The equation of continuity, together with (8), gives the first condition on the surface:

$$y = a, \qquad v'' + n^2 v = 0. \tag{9}$$

The stress normal to the surface is balanced by the capillary pressure. This leads to a second condition:

$$y = a, \qquad \pi - 2\eta v' = -\sigma\,\partial^2 a/\partial x^2, \tag{10}$$

where η is the viscosity and σ is the surface tension (the surface is assumed to be only slightly curved). Upon substitution of π from (10) into the first equation of (5) followed by differentiation with respect to time, this last condition becomes

$$y = a, \quad v''' + \left(\frac{ink}{\gamma} - \frac{inga^2}{2\gamma^2} - 3n^2\right)v' - \frac{i\delta n^3}{k\gamma}v = 0. \quad (11)$$

In Eq. (11), δ is the kinematic surface tension.

Without defining any new symbols, we introduce the dimensionless quantities

$$1 - y/a \to y, \quad na \to n, \quad ka/\gamma \to k. \quad (12)$$

If, furthermore, we have

$$R = ga^3/2\gamma^2, \quad S = \delta a/\gamma^2, \quad (13)$$

$$A = n(k - R), \quad B = nR, \quad C = n^3S/k, \quad (14)$$

then the Orr-Sommerfeld equation, together with its boundary conditions, becomes

$$v'''' + [(iA - 2n^2) + iBy^2]v'' - \{[i(n^2A + 2B) - n^4] + in^2By^2\}v = 0,$$

$$y = 1, \quad v = 0, \quad v' = 0,$$

$$y = 0, \quad v'' + n^2v = 0, \quad v''' + (iA - 3n^2)v' + iCv = 0. \quad (15)$$

We seek a solution of (15) in the form

$$v = \alpha_0 + \alpha_1 y + \alpha_2 y^2 + \alpha_3 y^3 + \alpha_4 y^4 + \alpha_5 y^5 + \ldots . \quad (16)$$

The series (16) can be cut off at the fifth power and the resulting fifth-order polynomial substituted into (15) and into the boundary conditions. The determinant of the resulting system of equations is then calculated and its real and imaginary parts set equal to zero, with the following result:

$$n^2A^3 + A^2B + (720 + 480n^2 + 18n^4 - 11n^6)A + 2(120 + 48n^2 - n^4)B - 4(120 + 24n^2 + n^4)C = 0,$$

$$(120 + 4n^2 - 11n^4)A^2 + 4(10 - n^2)AB - 8(12 + n^2)AC + 4B^2 - 16BC - 15(192 + 384n^2 + 64n^4 - n^8) = 0. \quad (17)$$

The next step should be to substitute for A, B, and C, to express S in terms of R, and to eliminate k. Then the curve R(n) would be the curve of equilibrium. However, this procedure is algebraically difficult, so that it becomes necessary to find the equilibrium curve numerically. Our calculations were carried out for water.

From (13) and the experimental data quoted in reference,[2] we have

$$S = \frac{\delta}{\gamma}\left(\frac{2}{g\gamma}\right)^{1/3} R^{1/3} = 3657\,R^{1/3}. \quad (18)$$

Considering (14) as a system of equations for n, k, and R, that is,

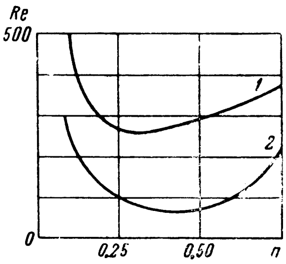
$$3657\,n^{11/3} = C(A + B)B^{-1/3}, \quad nk = A + B, \quad nR = B, \quad (19)$$

we assign arbitrary values to n. We choose a value of A to satisfy the first equation in (19) if B and C are obtained from (17). The other two equations determine k and R. From (1), (2), and (13) we find

$$\mathrm{Re} = \tfrac{8}{3}R. \quad (20)$$

The results are shown in the figure. Curve 1 corresponds to positive phase velocities, while curve 2 corresponds to negative ones. The phase velocities and wave lengths can be calculated from (12), (13), and (20).

Data applicable to the threshold of instability are shown in the table. It appears that the most "dangerous" perturbations are those traveling upward at 15.1 cm/sec. They break up the flow at a Reynolds number Re ~ 72, when wave conditions are produced.



	$k>0$	$k<0$	[1,2]
Re	260	72	22
$n=2\pi a/\lambda$	0.3	0.4	0.1
λ, cm	0.62	0.30	0.89
k, cm/sec	27.7	−15.1	12.4

At first glance it would appear that these conclusions contradict the data shown in the last column of the table, which are the results of P. L. Kapitza and have been verified experimentally.[1,2] However, the comparison must be made carefully. According to the experiments, the lower bound for the stability of wave flow is Re = 22. To establish this, artificial regular perturbations were introduced (photographs II, 6 — 12 in reference 2). Without these, wave flow occurs at a higher Reynolds number. The Reynolds numbers are easy to find from the data of photographs I, 1 — 2, and are 46, 55, and 54 respectively. The photographs do not show the volume rates of flow; however, these were presumably close to critical since the photographs must have been made as soon as wave flow was observed.

Wave flow appears and disappears at different

Reynolds numbers. Conditions for the existence of plane and wave flows are not mutually exclusive, but overlap partially. The process by which wave flow is established may be visualized as follows for the case when there are no artificial perturbations. As the rate of flow increases no waves appear as Re = 22 is reached. At Re = 72, waves will run upward at 15.1 cm/sec. As will appear shortly, these waves are not stable and will exist for only a short time. The plane flow will be disturbed, so that waves running downward can appear and establish stable wave flow. Wave flow will persist as the flow rate is decreased to Re = 22, when plane flow is again established.

The above considerations make it clear that there is no contradiction with the experimental data shown in the table. It would be desirable to measure the upper bound for stability of plane flow.

It is usually believed that P. L. Kapitza has theoretically established the transition between plane and wave flow and found it to occur in water at a Reynolds number of 21.6. In what follows we compare Kapitza's method with the method of small perturbations.

THE METHOD OF P. L. KAPITZA

1. Before proceeding with the comparison, we present a brief exposition of Kapitza's work,[1] using our notation.

Assume that the distribution of longitudinal velocity is the same in wave flow as it is in plane flow. The average (over y) velocity, together with its square, is substituted into the Navier-Stokes equation, without the third term on the left hand side:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g + \gamma\nabla^2 u. \quad (21)$$

The pressure is assumed to be constant over the cross section of the flow, and equal to the capillary pressure at the surface.

Taking the phase velocity in the profile of the liquid surface to be k, a constant, and writing the variable thickness of the layer in the form

$$a = a_0(1+\varphi), \quad (22)$$

P. L. Kapitza obtained the following equation in the second approximation:

$$\delta a_0 \dddot{\varphi}(1+3\varphi) + \overline{u}_0^2(z-1)(z-1.2)\left(1-\frac{z}{5z-6}\varphi\right)\dot{\varphi} + 3g\varphi^2 + 3\left(g - z\frac{\gamma\overline{u}_0}{a_0^2}\right)\varphi + \left(g - 3\frac{\gamma\overline{u}_0}{a_0^2}\right) = 0. \quad (23)$$

In this equation, a subscript zero denotes the middle cross section, a short horizontal bar over a symbol means an average over y, and the ratio of the phase velocity to the mean velocity has been written as z, the ratio being taken at the middle cross section.

The solution of (23) is

$$\varphi = \alpha\sin nx + 0.28\alpha^2\cos 2nx - \frac{g\alpha^2}{4n^3\delta a_0}\sin 2nx + \ldots \quad (24)$$

subject to the following supplementary conditions:

$$g = z\gamma\overline{u}_0 / a_0^2, \qquad g(1+3\alpha^2/2) = 3\gamma\overline{u}_0/a_0^2, \quad (25)$$

$$n^2 = (z-1)(z-1.2)\,\overline{u}_0^2/\delta a_0. \quad (26)$$

The amplitude α and the number z are determined from the equilibrium and energy-balance conditions. The fact that the average dissipated energy equals the work of the force of gravity leads to the relation

$$a_0^3 = 3\gamma QF/g,$$

$$F = \frac{1}{\lambda}\int_0^\lambda \frac{(1+z\varphi)^2}{(1+\varphi)^3}dx = \frac{1}{2}\{2 + \alpha^2[1 - 6z + z^2(1+2\alpha^2)]\}(1-\alpha^2)^{-5/2}. \quad (27)$$

As a function of the two variables z and α, the quantity F has a minimum at

$$\alpha = 0.46, \qquad z = 2.4 \quad (28)$$

The mean potential energy associated with the surface tension is

$$\overline{E}_\sigma = \frac{1}{4}\sigma a_0^2\alpha^2 n^2[1 + 4(g\alpha/4a_0\delta)^2 n^{-6} + \ldots] \quad (29)$$

and from this expression the critical value for Re can be found as follows:

Imagine wave flow to be established. As long as the quantity in the brackets is small, the surface energy will decrease with increasing λ and decreasing n. However, for some critical value of λ the part in brackets will become large and the energy will begin to grow. Such waves are presumably unstable and so the wave flow will disappear. Hence the condition

$$\partial\overline{E}_\sigma/\partial n = 0 \quad (30)$$

defines the limiting wave length beyond which wave flow is unstable. From (26), (27), (29), and (30) it follows that

$$\lambda/a_0 = 13.5Q/\gamma, \quad \mathrm{Re} = 2.43(\delta^3/g\gamma^4)^{1/11} = 0.3\lambda/a_0, \quad (31)$$

For water this yields Re = 21.6.

The above method, due to P. L. Kapitza, gives the lower boundary for stability of wave flow, while the method of small perturbations gives the upper boundary for stability of plane flow. The results obtained by use of these two methods are not contradictory.

2. Kapitza's results have been experimentally verified. However, the following comments can be made about the theory, as summarized above.

First, the longitudinal velocity cannot be distributed parabolically in wave flow, for if it were, then the equation of continuity would imply a cubically-distributed transverse velocity. The Orr-Sommerfeld equation (15) would then be satisfied by a cubic polynomial (16), which in fact is not the case. Unfortunately, it is difficult to see what this fundamental assumption could be replaced by. Furthermore, in the Navier-Stokes equation the quantities u and u^2 should be the true velocities, not mean values. It is the whole equation that should be averaged over y. The third term on the left hand side of the Navier-Stokes equation is of the same order of magnitude as the second term and so cannot be discarded (this follows from the equation of continuity). On the other hand, derivatives with respect to x can be neglected in comparison with derivatives taken with respect to y. Hence the equations to be considered are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g + \gamma\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{32}$$

To avoid a complicated analysis of both equations, we must take the pressure to be constant over the cross section. This is only an approximation, since without a transverse pressure gradient there can be no transverse velocity and hence no wave flow.

It is difficult to find the minimum of the function F. It is easier to consider F as a function of one variable. From (25) we obtain

$$\alpha^2 = 2(3-z)/z. \tag{33}$$

From this it follows that waves with $z < 0$, i.e., those traveling upward, are impossible.

3. If the above comments are accepted, then Kapitza's method can be developed more rigorously. From (32) we obtain in lieu of (23)

$$3\delta a_0 \varphi\dddot{\varphi} + \delta a_0 \dddot{\varphi} - \frac{2}{5} z^2 \overline{u_0^2} \varphi\dot{\varphi} + \frac{1}{5}(5z^2 - 12z + 6)\overline{u_0^2}\dot{\varphi}$$
$$+ 3g\varphi^2 + 3(g - z\gamma\overline{u_0}/a_0^2)\varphi + (g - 3\gamma\overline{u_0}/a_0^2) = 0. \tag{34}$$

Conditions (25) and (33) still hold. The solution

$$\varphi = \alpha\sin nx - \frac{g\alpha^2}{4n^3\delta a_0}\sin 2nx$$
$$+ \frac{17z^2 - 36z + 18}{12(5z^2 - 12z + 6)}\alpha^2\cos 2nx + \cdots \tag{35}$$

leads to the following improved approximation to the wave number

$$n^2 = \frac{1}{5}(5z^2 - 12z + 6)(\overline{u_0^2}/\delta a_0). \tag{36}$$

Considering the integrand in (27) to be a function of φ, we expand it in a power series and keep no terms higher than quadratic. Using (35) and (33) we find

$$F = 1 + (3-z)(z^2 - 6z + 6)/3z, \tag{37}$$

and hence upon comparing (27) with the first equation of (25) we obtain

$$F = z/3. \tag{38}$$

From (37) and (38) we obtain

$$(3-z)(z^2 - 5z + 6) = 0. \tag{39}$$

One root of this equation corresponds to wave flow and leads, instead of (28), to

$$z = 2, \qquad \alpha = 0.58. \tag{40}$$

The remaining two equal roots give plane flow ($z = 3$, $\alpha = 0$).

It was not necessary to find the minimum of F because the stable solution had already been found in (35). However, the same result could be obtained by finding the minimum of the function (37).

For comparison with experiment, it is not difficult to find, from (25), (36) and (40), parameters such as wave length, phase velocity, maximal longitudinal velocity (the rate at which a dye would spread), mean thickness of the layer, amplitude, and the thickness of the plane flow which would give the same volume rate of flow. All these quantities, as functions of the volume rate of flow, differ but little from those obtained by Kapitza, so that they can be thought of as verified by experiment.[2] The biggest discrepancy with experiment occurs for the amplitude; however, the authors themselves do not consider the experimental values for the amplitude to be a good test of the theory.

The situation is quite different for the critical value of Re. Let us compute it using Kapitza's method. Substituting (40) into (35) we find

$$\varphi = 0.58\Big(\sin nx - 0.145\frac{g}{n^3\delta a_0}\sin 2nx$$
$$+ 0.338\cos 2nx + \cdots\Big). \tag{41}$$

From (41), the mean surface energy can be calculated, as follows:

$$\overline{E}_\sigma = \sigma\,\overline{[1 + (a_0\dot{\varphi})^2]^{1/2}} \tag{42}$$

so that, computing the derivative (30), we obtain

$$g/n^3\delta a_0 \leqslant 2.94. \tag{43}$$

From (36), (27) and (38) it follows that

$$n^2 = 0.2gQ/\gamma\delta, \quad \lambda = 14.1(\gamma\delta/gQ)^{1/2}, \quad a_0 = 1.26(\gamma Q/g)^{1/2}. \tag{44}$$

Substituting these into (43), instead of (31), it turns out that

$$\mathrm{Re} = 1.08\lambda / a_0 = 7.32(\delta^3 / g\gamma^4)^{1/11}. \qquad (45)$$

For water, the data of reference 2 give

$$\mathrm{Re} = 64. \qquad (46)$$

It thus appears that Kapitza's value of the lower limit of stable wave flow disagrees with experiment. This is not surprising in view of the comments made above.

CONCLUSION

The methods described above, together with the results obtained by their use, are first attempts to understand a new, important, problem in hydrodynamic stability. Both methods could be developed further, though this would be accompanied by great computational difficulties. The method of small perturbations could be improved by taking more terms in the series (16), or by solving the Orr-Sommerfeld equation differently. It would be interesting to apply Kapitza's method to the complete system of hydrodynamic equations for this problem.

The author would like to thank V. S. Sorokin for suggesting this problem and S. P. Kapitza for useful discussions.

[1] P. L. Kapitza, JETP **18**, 3, 19 (1948).

[2] P. L. Kapitza and S. P. Kapitza, JETP **19**, 105 (1949).

[3] C. C. Lin, The Theory of Hydrodynamic Stability, Cambridge, 1955.

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