ANGULAR DISTRIBUTION OF FISSION FRAGMENTS PRODUCED BY LOW-ENERGY NEUTRONS

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The angular distribution of fission fragments produced in the capture of particles of low orbital angular momentum is considered. The spin of the target nucleus is taken into account. The effect of fluctuations in the distribution of the transition nucleus levels on the angular distribution of the fission fragments is also considered.

UNTIL recently, the available data on the angular distribution of fission fragments referred chiefly to fission induced by high-energy particles. Abundant experimental data for this energy region made it possible to establish the existence of not only qualitative but also quantitative agreement between the simple statistical theory of the angular anisotropy and experiment.[1-4]

The investigation of the low-energy excitation region of the transition nucleus* is of considerable interest. The study of the angular distribution of the fragments permits one to obtain information on the fission probability as a function of the value of the projection K of the spin of the transition nucleus on the direction of fission, and therefore on the K distribution of the transition nucleus levels for small excitation. When the transition nucleus is even — even, the fragment angular distribution also gives additional evidence of the existence of an energy gap in the spectrum of the transition nucleus levels and the existence of rotational levels inside the gap. However, there is need of a more accurate theory of the fragment angular distribution. Most important is the taking into account of the initial spin of the nucleus.

1. For a definite spin of the target nucleus, the probability amplitude for the emission of fragments in the direction **n** can be written in the form[5]

$$f_{lSJ}(\mathbf{n}) = \sqrt{\frac{4\pi}{2(2J_0+1)}} \sum_{m\mu M} C^{JM}_{S\mu lm} Y_{lm}(\mathbf{\nu}) \sum_{K=-J}^{J} \sqrt{p_J(K)}\, D^J_{MK}(\mathbf{n}), \quad (1)$$

where S is the spin of the channel; $S = J_0 \pm 1/2$; J_0 is the spin of the target nucleus; J is the spin of the compound nucleus; l is the neutron orbital angular momentum; ν is the direction of the neutron beam; m, μ, M are the projections of the vectors l, S, J on the axis of quantization; Y_{lm} and D^J_{MK} are the normalized spherical function and the matrix of the J-representation of the group of rotations; $C^{c\gamma}_{a\alpha b\beta}$ is the Clebsch — Gordan coefficient. Choosing in (1) the axis of quantization along the direction of fragment emission **n**, one can obtain the following expression for the amplitude:

$$f_{lSJ} = \sqrt{(2J+1)/2(2J_0+1)} \sum_{m\mu} C^{JK}_{S\mu lm} \sqrt{p_J(K)}\, Y_{lm}(\mathbf{\nu}), \quad (2)$$

where use was made of the relation

$$D^J_{MK}(0) = \{(2J+1)/4\pi\}^{1/2}\, \delta_{MK}.$$

For the fragment angular distribution, we obtain

$$W_{lSJ}(\vartheta) = \{(2J+1)/2(2J_0+1)\} \times \sum_{m\mu} (C^{JK}_{S\mu lm})^2\, p_J(K)\, |Y_{lm}(\vartheta)|^2, \quad (3)$$

where ϑ is the angle between the direction of the neutron beam and the direction of fission. In formula (3) the interference terms have been discarded, as they are not important for a nucleus with a large density of levels.

The coefficients $p_J(K)$ are determined by the number of transition nucleus levels, for a given value of K, through which the fission takes place. They can be represented in the form

$$p_J(K) = \{\Gamma_f(0)\, a(K)/(2J+1)\} \times \Big\{\Gamma_n + \Gamma_\gamma + \Big[\Gamma_f(0) \sum_{K=-J}^{J} a(K)\Big] \Big/ (2J+1)\Big\}^{-1}, \quad (4)$$

where a(K) is independent of J, and $\Gamma_f(0)$ is the fission width for J = 0. [It is assumed that a(K) is normalized by the condition a(0) = 1]. With

*By transition nucleus we have in mind here a nucleus undergoing fission with a deformation corresponding approximately to a saddle point. The excitation energy of such a nucleus is approximately equal to the excitation energy of a compound nucleus minus the threshold energy for fission.

such a choice of $p_J(K)$, the angular distribution (3) turns out to be normalized in such a way that the integral of $W_{lSJ}(\vartheta)$ is equal to the probability of fission for the nuclear angular momentum J if the absorption coefficient is equal to unity:

$$\int W_{lSJ}(\vartheta)\,d\Omega = (2J+1)\Gamma_f/\Gamma; \quad (5)$$

$$\Gamma = \Gamma_n + \Gamma_\gamma + \Gamma_f(J), \quad \Gamma_f(J) = (2J+1)^{-1}\Gamma_f(0)\sum_{K=-J}^{J} a(K), \quad (6)$$

Γ_f is the fission width in the state J, M.* In formula (5), 2J + 1 is the statistical factor for the formation of a compound nucleus of angular momentum J and any allowed value of M for given l and S.

We now assume that the total width Γ of the excited state of the nucleus weakly depends on the angular momentum. This occurs, in particular, if $\Gamma_n + \Gamma_\gamma$ does not depend on J, and $\Gamma_f(J) < \Gamma_n + \Gamma_\gamma$ and also when $\sum_{K=-J}^{J} a(K) \approx 2J+1$, i.e., if $a(K)$ is very little different from unity. Taking this into account, we find the following expression for the angular distribution:

$$W_{lSJ}(\vartheta) = \sum_{\mu m K}{}' (C^{JK}_{S\mu\, lm})^2\, a(K)\, |Y_{lm}(\vartheta)|^2. \quad (7)$$

In formula (7) we have omitted factors which are independent of l and J and which are not important for what follows.

The overall angular distribution of the fragments in the reaction has the form

$$W(\vartheta) = \sum_{lSJ} \zeta_{lSJ} W_{lSJ}(\vartheta), \quad (8)$$

where ζ_{lSJ} are the absorption coefficients. For the sake of simplicity, we assume that they do not depend on S and J. Setting $\zeta_{lSJ} \approx \zeta_l$ we find the following expression for the angular distribution of the fragments:

$$W(\vartheta) = \sum_l \zeta_l \sum_{S=J_0\pm 1/2} W_{lS}(\vartheta), \quad (9)$$

where $W_{lS}(\vartheta)$ is the angular distribution for the channel (l, S):

$$W_{lS}(\vartheta) = \sum_{J=|l-S|}^{l+S} W_{lSJ}(\vartheta) = \sum_{m=-l}^{l} |Y_{lm}(\vartheta)|^2 \sum_{K=m-S}^{m+S} a(K). \quad (10)$$

In the derivation of formula (10) we employed the relation

$$\sum_J (C^{JK}_{S\mu\, lm})^2 = \begin{cases} 1, & |K-m| \leqslant S \\ 0, & |K-m| > S \end{cases}.$$

In statistical theory the K distribution is given by the expression[1,7]

$$a(K) = \exp(-K^2/2K_0^2), \quad (11)$$

where $K_0^2 = \mathcal{J}T/\hbar^2$; T is the transition nucleus temperature, $\mathcal{J}^{-1} = \mathcal{J}_{\parallel}^{-1} - \mathcal{J}_{\perp}^{-1}$, $\mathcal{J}_{\parallel}$ and $\mathcal{J}_{\perp}$ are the moments of inertia of the transition nucleus with respect to the axis of symmetry (axis of fission) and the direction perpendicular to it. Taking into account the fact that for low angular momenta the inequality

$$J^2/2K_0^2 \ll 1, \quad (12)$$

holds, we expand $a(K)$ into the series

$$a(K) \approx 1 - K^2/2K_0^2 + \ldots . \quad (13)$$

Inserting this expression into formulas (9) and (10), we obtain

$$W(\vartheta) \approx \sum_l \zeta_l \sum_{m=-l}^{l} |Y_{lm}(\vartheta)|^2 \sum_S (2S+1)$$
$$\{1 - (1/2K_0^2)[m^2 + S(S+1)/3]\}$$
$$= \frac{1}{4\pi}\sum_l (2l+1)\zeta_l \{1 - [l(l+1)/4K_0^2]\sin^2\vartheta\} + \text{const}, \quad (14)$$

which can also be represented in the form

$$W(\vartheta) \approx \text{const}\cdot\{1 - (\overline{l^2}/4K_0^2)\sin^2\vartheta\}, \quad (15)$$

where

$$\overline{l^2} = \Big\{\sum_l (2l+1)\zeta_l l(l+1)\Big\}\Big\{\sum_l (2l+1)\zeta_l\Big\}^{-1} \quad (16)$$

is the mean square of the orbital angular momentum imparted to the nucleus. For a black nucleus

$$\overline{l^2} = \tfrac{1}{2}\, l_{max}^2 = \tfrac{1}{2}(k_n R)^2. \quad (17)$$

For neutrons and nuclei with $A \sim 240$, the following expression is, in practice, more accurate:

$$\overline{l^2} \approx (2.5-3)\cdot E_n \ [\text{Mev}].$$

The small constant term dependent on the spin of the target nucleus and appearing in the braces in formula (14) has been omitted, since it obviously does not affect the shape of the angular distribution. It may therefore be said that for a small anisotropy, the statistical distribution of the fragments does not depend on the nuclear spin for any value of the orbital angular momentum of the neutron. The dependence of the angular distribution on the spin arises for the next term in the series (13). In the calculation of this term, the coefficient of $\sin^2\vartheta$ in formula (15) should be multiplied by

*Here the dependence of the fission width on the angular momentum of the nucleus, which is associated with the effect of rotation on the value of the fission threshold,[6] is not taken into account. This effect is important for very large angular momenta of the nucleus.

the spin-dependent factor $1 - S^2/2K_0^2$. The coefficient of $\sin^4 \vartheta$ is independent of the spin. For a small anisotropy, the spin-dependent terms are negligibly small.

The reason for the weak dependence of the anisotropy on the value of the initial spin becomes more understandable if one considers the classical limit (l, $S \gg 1$). In this approximation, for a given distribution $a(K)$ the angular distribution of the fragments with respect to the spin direction of the compound nucleus can be represented in the form

$$W(\mathbf{n}) = \text{const} \cdot a(K)|_{K=\mathbf{Jn}}. \qquad (18)$$

We set $\mathbf{J} = \mathbf{l} + \mathbf{S}$ and average expression (18) over all possible directions of the vectors $\mathbf{l}$ and $\mathbf{S}$ (vector $\mathbf{l}$ lies in a plane perpendicular to the beam, vector $\mathbf{S}$ lies on a sphere). As a result, we obtain for the Gaussian distribution (13) an expression similar to formula (15) and, consequently, a weak dependence of the angular distribution on the initial spin. For another form of $a(K)$, the angular distribution, generally speaking, depends on the initial spin. In particular, a strong dependence on the magnitude of the initial spin occurs for the distribution used by Griffin:[4]

$$a(K) = \begin{cases} 1 - |K| / K_{max}, & |K| \leqslant K_{max} \\ 0, & |K| > K_{max} \end{cases}$$

(the anisotropy is less for a large value of S). If the series for $a(K)$ begins with the fourth power of K

$$a(K) \approx 1 - \varkappa K^4 \qquad (\varkappa > 0),$$

then the angular distribution has the form

$$W(\vartheta) = \text{const} \cdot \left\{1 - \varkappa\left(\tfrac{3}{8}\,\bar{l}^4 \sin^4 \vartheta + S^2 \bar{l}^2 \sin^2 \vartheta\right)\right\},$$

i.e., the anisotropy would even increase with an increase in the initial spin. The physical reason for this is the fact that along with the deterioration in the angular momentum of the compound nucleus for a definite initial spin, there is an effect leading to an increase in the anisotropy; this effect is connected with the fact that for a large initial spin there are, on the average, larger values of the angular momentum of the compound nucleus (see also reference 1). Therefore, the final result is determined by the degree of the dependence of the anisotropy on the magnitude of the angular momentum of the nucleus.

2. One can also set the task of determining the distribution $a(K)$ from the known angular distribution of the fragments. To do this, it is necessary to know the angular distribution for each K, separately.

We shall first consider the case $K = 0$. Since it is assumed that the transition nucleus has an axis of symmetry (coinciding with the direction of fission), there exists for the rotational states with $K = 0$ a selection rule, according to which only even values of angular momentum can occur for a compound nucleus of positive parity and odd values, for negative parity. Hence, for $K = 0$, the summation over J in the individual terms of formula (8) should be carried out in such a way that the parity of J is the same as the parity of l for a target nucleus of positive parity and of the opposite parity in the case of negative parity. This summation can be carried out in general form by means of the formula

$$\sum_J{}' (C^{J0}_{S\mu l m})^2 = \tfrac{1}{2}[1 + (-1)^{l+S+J}\delta_{m0}], \qquad (19)$$

where the primes on the summation sign indicate that the summation is carried out only over even or only over odd values of J. Employing (8) and (19), we find

$$W^{(K=0)}(\vartheta) = \sum_{lS} \zeta_l W^{(K=0)}_{lS}(\vartheta), \qquad (20)$$

$$W^{(K=0)}_{lS}(\vartheta) = \tfrac{1}{2}\left\{\sum_{|m|\leqslant S} |Y_{lm}(\vartheta)|^2 + (-1)^{l+S+J} |Y_{l0}(\vartheta)|^2\right\}. \qquad (21)$$

For each l the second term in formula (20) has opposite signs for the two values $S = J_0 \pm \frac{1}{2}$, and these terms cancel one another in the sum over S in formula (20). This also occurs in the case of a spin-orbit interaction between the neutron and nucleus when the absorption coefficient also depends on S and J. Therefore we shall omit this term everywhere below in the expressions for $W^{(K=0)}_{lS}(\vartheta)$.

For $l \leq S$

$$W^{(K=0)}_{lS}(\vartheta) = (2l + 1)/8\pi, \qquad (22)$$

i.e., for $K = 0$ the channels with $l \leq S$ give an isotropic contribution to the angular distribution. For the channels with $l > S$, the angular distribution for $K = 0$ can be written in the form

$$W^{(K=0)}_{lS}(\vartheta) = \tfrac{1}{2} \sum_{|m|\leqslant S} |Y_{lm}(\vartheta)|^2 = \frac{2l+1}{8\pi} F_{lS}(\vartheta), \qquad (23)$$

$$F_{l\lambda}(\theta) = \frac{4\pi}{2l+1} \sum_{m=-\lambda}^{\lambda} |Y_{lm}(\vartheta)|^2. \qquad (24)$$

A simple expression for the function $F_{l\lambda}$ is obtained in the quasi-classical approximation if, instead of the functions $|Y_{lm}|^2$, their classical analog

$$(1/2\pi^2)\{\sin^2 \vartheta - (m^2/l^2)\}^{-1/2} \qquad (25)$$

is employed. Replacing in formula (24) the summation over m by an integration, we obtain

$$F_{l\lambda}(\vartheta) \approx \begin{cases} 1 & l \sin\vartheta \leqslant \lambda \\ (2/\pi)\sin^{-1}(\lambda/l\sin\vartheta), & l\sin\vartheta \geqslant \lambda. \end{cases} \quad (26)$$

For $\lambda \ll l$

$$F_{l\lambda}(\vartheta) \sim (2/\pi)(\lambda/l\sin\vartheta).$$

By means of formula (10), one can obtain expressions for the angular distribution for other fixed values of K. Noting that a(K) is an even function of K, we obtain the following resulting expression for the angular distribution of the fission fragments for a channel with a fixed value of |K|:

$$W_{lS}^{(|K|)}(\vartheta) = W_{lS}^{(K)}(\vartheta) + W_{lS}^{(-K)}(\vartheta),$$

where for $S \geq l$

$$W_{lS}^{(|K|)}(\vartheta) = \frac{2l+1}{4\pi}\begin{cases} 1 & |K| \leqslant S-l \\ 1 + F_{l,S-|K|}(\vartheta), & S-l < |K| \leqslant S, \\ 1 - F_{l,|K|-S-1}(\vartheta), & S < |K| \leqslant S+l, \end{cases} \quad (27)$$

and for $S < l$

$$W_{lS}^{(|K|)}(\vartheta) = \frac{2l+1}{4\pi}\begin{cases} F_{lS}(\vartheta), & K=0 \\ F_{l,S-|K|}(\vartheta) + F_{l,S+|K|}(\vartheta), & 0 < |K| \leqslant S \\ 1 - F_{l,|K|-S-1}(\vartheta), & S < |K| \leqslant l+S. \end{cases} \quad (28)$$

Substituting (27) and (28) into formulas (9) and (10), we obtain the following expression for the angular distribution:

$$W(\vartheta) = \text{const}\cdot\sum_l (2l+1)\bar{\zeta}_l \cdot [a(S_1-\lambda) - a(S_1+\lambda)](F_{l,\lambda-1} + F_{l,\lambda}),$$

$$S_1 = J_0 + 1/2, \qquad a(-K) = a(K). \quad (29)$$

We introduce the quantity $A(\vartheta) = \{\sigma_f(\vartheta) - \sigma_f(90°)\}/\sigma_f(0°)$. For a small anisotropy, $A(\vartheta)$ practically coincides with the usually considered quantity $\{\sigma_f(\vartheta) - \sigma_f(90°)\}/\sigma_f(90°)$. The expression for $\sigma_f(0)$ can be obtained directly from formulas (9) and (10) if it is noted that

$$|Y_{lm}(0)|^2 = \{(2l+1)/4\pi\}\,\delta_{m0}.$$

Calculating $\sigma_f(0°)$ in this way, we find

$$A(\vartheta) = \sum_{\lambda=1} \beta_\lambda \Phi_\lambda(\vartheta), \quad (30)$$

where

$$\beta_\lambda = \frac{a(S_1-\lambda) - a(S_1+\lambda)}{2\{a(0) + a(S_1) + 2[a(1) + \ldots + a(S_1-1)]\}} \quad (J_0 \neq 0); \quad (31)$$

$$\Phi_\lambda(\vartheta) = \left\{\sum_l (2l+1)\bar{\zeta}_l \Phi_{l\lambda}(\vartheta)\right\} \Big/ \sum_l (2l+1)\zeta_l,$$

$$\Phi_{l\lambda}(\vartheta) = F_{l,\lambda-1}(\vartheta) + F_{l,\lambda}(\vartheta) - F_{l,\lambda-1}(90°) - F_{l,\lambda}(90°). \quad (32)$$

For a small anisotropy, the denominator in formula (31) is approximately equal to $4S_1 = 2(2J_0+1)$. Some values of the functions $\Phi_{l\lambda}$ for $l \leq 3$ are shown in the table.

The functions $\Phi_\lambda(\vartheta)$ depend only on the incident neutron energy, since they are fully determined by the absorption coefficients. In the case of a black nucleus, they depend only on l_{max}. The index λ runs over the values from 1 to l_{max}. Thus, knowing the angular distribution of the fragments, one can, in principle, determine l_{max} independent differences of the coefficients a(K). Using the functions (26), we can obtain the following quasiclassical expression for $\Phi_\lambda(\vartheta)$ for a black nucleus:

$$\Phi_\lambda(\vartheta) = \sum_{n=\lambda-1}^{\lambda} \bar{F}_{ln}\Big|_{90°}^{\vartheta},$$

$$\bar{F}_{ln}(\vartheta) \approx \begin{cases} 2\pi^{-1}(\sin^{-1}x + x\sqrt{1-x^2}), & x = n/l_{max}\sin\vartheta \leqslant 1 \\ 1, & x > 1 \end{cases}.$$

The angular distribution of the fragments is readily calculated by means of formulas (30) — (32) if the distribution of a(K) is given. If, as happens in the statistical case, a(K) is a monotonically decreasing function of K, all the coefficients β_λ are positive, and $A(\vartheta)$ is a monotonically decreasing function of the angle ϑ. [Both functions $\Phi_\lambda(\vartheta)$ and $\Phi_{l\lambda}(\vartheta)$ decrease monotonically with an in-

Values of the function $\Phi_{l\lambda}$

ϑ, deg	l, λ					
	1.1	2.1	3.1	2.2	3.2	3.3
0	1.000	1.500	1.636	0.750	1.272	0.636
10	0.970	1.412	1.460	0.750	1.267	0.636
20	0.884	1.170	1.032	0.740	0.226	0.634
30	0.750	0.844	0.556	0.703	1.076	0.626
40	0.589	0.518	0.218	0.623	0.810	0.593
50	0.413	0.256	0.066	0.492	0.486	0.510
60	0.250	0.062	0.036	0.330	0.218	0.373
70	0.117	0.020	0.034	0.165	0.068	0.205
80	0.030	0.002	0.020	0.041	0.024	0.065
90	0.000	0.000	0.000	0.000	0.000	0.000

crease in ϑ.] In the statistical case, for $a(K)$ not very different from unity, one can employ expansion (13), which gives

$$\beta_\lambda \approx \beta_\lambda^0 = \lambda / 2K_0^2. \quad (33)$$

The function $A(\vartheta)$, by (15), has the form

$$A(\vartheta) = (1/2K_0^2) \sum_\lambda \lambda \Phi_\lambda(\vartheta) = (\overline{l^2}/4K_0^2)\cos^2\vartheta. \quad (34)$$

In formulas (26) — (31), the above-mentioned selection rule for J was not taken into account for the states with K = 0. If this rule is taken into account, the coefficient $a(0)$ decreases by one-half. This result is obvious in the classical limit: For a given orbital angular momentum and K = 0, fission is possible for only half the possible values of the angular momentum of the compound nucleus. As a result, formula (30) should be written in the form

$$A(\vartheta) \approx \sum_{\lambda=1} \beta_\lambda \Phi_\lambda(\vartheta) + \tfrac{1}{2}[2(2J_0+1)]^{-1}\Phi_{S_1}(\vartheta),$$

$$(S_1 \neq 1/2, \quad l_{\max} \geqslant S_1, \quad a(0) = 1). \quad (34')$$

For a Gaussian distribution of $a(K)$ and a small anisotropy, the fragment angular distribution in which the selection rule for the states with K = 0 is taken into account has the form

$$A(\vartheta) \approx (\overline{l^2}/4K_0^2)\cos^2\vartheta - \tfrac{1}{2}[2(2J_0+1)]^{-1}\Phi_{S_1}(\vartheta). \quad (35)$$

For large l the integral contribution of the second term decreases as $l_{\max}^{-1}$. If the first term in formula (35) is independent of the initial spin, then the second term, other conditions being equal, leads to a certain decrease in the anisotropy for a smaller value of J_0.

3. In those cases for which the probability of fission in a channel of angular momentum J is determined not only by some statistical factors, as was assumed above, there may be need of expressions for the fragment angular distribution with a fixed value of J. In the classical limit, they can be obtained by replacing in formula (7) the square of the Clebsch-Gordan coefficient by its classical analog:

$$(C_{S\mu lm}^{J0})^2 \approx (1/\pi l)\{\sin^2\chi - (m^2/l^2)\}^{-1/2},$$

$$\cos\chi = (2lJ)^{-1}[l(l+1) - S(S+1) + J(J+1)], \quad (36)$$

and the functions $|Y_{lm}(\vartheta)|^2$ by expression (25). Replacing also the summation over m by integration, we obtain

$$W_{lSJ}^{(K=0)}(\vartheta) = \pi^{-3}\begin{cases}(\sin\chi)^{-1}K(z), & z = \sin^2\vartheta/\sin^2\chi < 1\\ (\sin\vartheta)^{-1}K(z^{-1}), & z \geqslant 1\end{cases}, \quad (37)$$

where $K(z)$ is the complete elliptical integral of the first kind:

$$K(z) = \int_0^{\pi/2} (1 - z^2\sin^2\varphi)^{-1/2}\, d\varphi.$$

The quantum-mechanical expression for $W_{lSJ}^{(K=0)}$ has the form

$$\sum_\rho (-1)^S (2l+1)(2J+1)(C_{l0l0}^{\rho 0})^2$$

$$\times (C_{J0J0}^{\rho 0})^2 W(lJlJ \mid S\rho) P_\rho(\cos\vartheta), \quad (38)$$

where $W(abcd \mid ef)$ are Racah coefficients.

Another classical expression for $W_{lSJ}^{(K=0)}(\vartheta)$ can be obtained by replacing the Racah coefficients in (38) and the squares of the Clebsch-Gordan coefficients by their classical analogs:

$$(C_{l0l0}^{\rho 0})^2 \approx (4/\pi)\{4l^2 - \lambda^2\}^{-1/2},$$

$$W(lJlJ \mid S\rho) \approx (-1)^{S-l-J}\{(2l+1)(2J+1)\}^{-1/2} P_\rho(\cos\chi).$$

As a result, we obtain

$$W_{lSJ}^{(K=0)}(\vartheta) \approx \frac{8}{\pi}\sum_\rho \{(2l+1)(2J+1)^{1/2}\{(4J^2 - \rho^2)$$

$$\times (4l^2 - \rho^2)\}^{-1/4} P_\rho(\cos\chi) P_\rho(\cos\vartheta). \quad (39)$$

The last expression for $W_{lSJ}^{(K=0)}(\vartheta)$ is somewhat more accurate than (37).

4. In the presence of a spin-orbit interaction between the incident neutron and nucleus, the absorption coefficients ζ_{lSJ} depend on all the indices. Their explicit form can be found from a comparison of the expressions for the total wave function of the system in the representation used above for the spin of the channel and the ordinary j-representation ($j = l \pm 1/2$). Consideration of the spin-orbit interaction leads primarily to the replacement of ζ_l by the mean absorption coefficient

$$\overline{\zeta}_l = (2l+1)^{-1}\{(l+1)\zeta_{l,l+1/2} + l\zeta_{l,l-1/2}\},$$

where $\zeta_{l,j}$ are the absorption coefficients in the j-representation. In formulas (15), (34), and (35) there appears, moreover, the factor

$$[1 + 2(2l+1)^{-1}q_l],$$

$$q_l = (\zeta_{l,l+1/2} - \zeta_{l,l-1/2})/(\zeta_{l,l+1/2} + \zeta_{l,l-1/2})$$

in the coefficients of $\cos^2\vartheta$. This correction is not important. For $W^{(K=0)}(\vartheta)$, with allowance for the spin-orbit interaction, we obtain the expression

$$W^{(K=0)}(\vartheta) = \tfrac{1}{2}\Big\{\sum_{l,S} \overline{\zeta}_l \sum_{m=-S}^{S} |Y_{lm}(\vartheta)|^2$$

$$- \sum_{l \geqslant S_1} \overline{\zeta}_l [4q_l S_1/(2l+1)][1 + q_l/(2l+1)]\,|Y_{l,S_1}(\vartheta)|^2\Big\}. \quad (23')$$

The correction associated with the last term in (23′) also proves to be small. We note the following useful relations which are employed in calculations with spin-orbit coupling:

$$\sum_J J(J+1)(C^{JK}_{S\mu lm})^2 = \begin{cases} l(l+1)+S(S+1)+2m\mu, & |\mu|\leqslant S, \ |m|\leqslant l \\ 0 \quad , & |\mu|>S, \ |m|>l \end{cases},$$

$$\sum{}' J(J+1)(C^{J0}_{S\mu lm})^2 = \tfrac{1}{2}[1+(-1)^{S+l+J}\delta_{m0}][l(l+1)+S(S+1)]-m^2.$$

The validity of these relations, as well as formula (19), can be established by a comparison of the zero coefficients for ϑ^0, ϑ^2, $(\pi-\vartheta)^0$, and $(\pi-\vartheta)^2$ in the expansion in powers of ϑ and $(\pi-\vartheta)$ of the identity

$$D^S_{\mu\mu}D^l_{mm} - \sum_J (C^{JK}_{S\mu lm})^2 D^J_{KK} = 0.$$

5. If the values of the orbital angular momenta taking part in the reaction are not large, formula (30) can be used quite directly for the analysis of the experimental data. The "statistical" term of the angular distribution, which, for a small anisotropy, is given by formula (34), can be separated immediately. With the "fluctuational" part of the angular distribution separated in this way, it can readily be shown how the coefficients β_λ differ from their statistical values (33). To do this, it is necessary only to take into account the fact that the functions $\Phi_\lambda(\vartheta)$, which, although simple and similar in form, strongly differ as regards the width of the maxima (see Fig. 1). It is also important that the functions $\Phi_\lambda(\vartheta)$ very weakly depend on the assumptions concerning the absorption coefficients ζ_l. This can be seen in Fig. 1, where $\Phi_\lambda(\vartheta)$ calculated for two variants of $\overline{\zeta}_l$ are shown. The dotted curve denotes Φ_λ for a black nucleus with $l_{max} = 3$ ($\zeta_l = 1$, $l \le l_{max}$). The solid curve represents the functions Φ_λ calculated with values of ζ_l computed by Nemirovskiĭ for a semi-transparent spherical nucleus with a diffuse boundary for KR = 11.5 and $E_n = 1.5$ Mev ($\overline{\zeta}_0 = 0.45$, $\overline{\zeta}_1 = 0.95$, $\overline{\zeta}_2 = 0.29$, $\overline{\zeta}_3 = 0.50$).* The nonmonotonic character of $\overline{\zeta}_l$ reflects the existence of a "resonance shape" for the p and f waves. The effect of the resonances decreases, owing to the deformation of the target nucleus, as a result of which the true values of the functions Φ_λ prove to be intermediate between the two curves shown in Fig. 1, which, however, are very close to each other.

The angular distribution of the fission fragments has been studied experimentally by Blumberg and Leachman[8] and by Henkel and Simmons.[9] Comparison of the angular distributions for U^{233} ($J_0 = 5/2$; fission threshold of 1.6 Mev), Pu^{239} ($J_0 = 1/2$, threshold of 1.6 Mev) and U^{235} ($J_0 = 7/2$, threshold of 0.6 Mev) indicates the absence of an appreciable influence of the spin of the target nucleus. Moreover, according to the experimental data of Henkel and Simmons,[9] the angular distributions for these targets are described, within the limits of experimental accuracy, by the expression $1 + A\cos^2\vartheta$, where $A \approx 0.1 - 0.15$. Both these circumstances are evidence in favor of a Gaussian distribution for a(K).* On the other hand, the

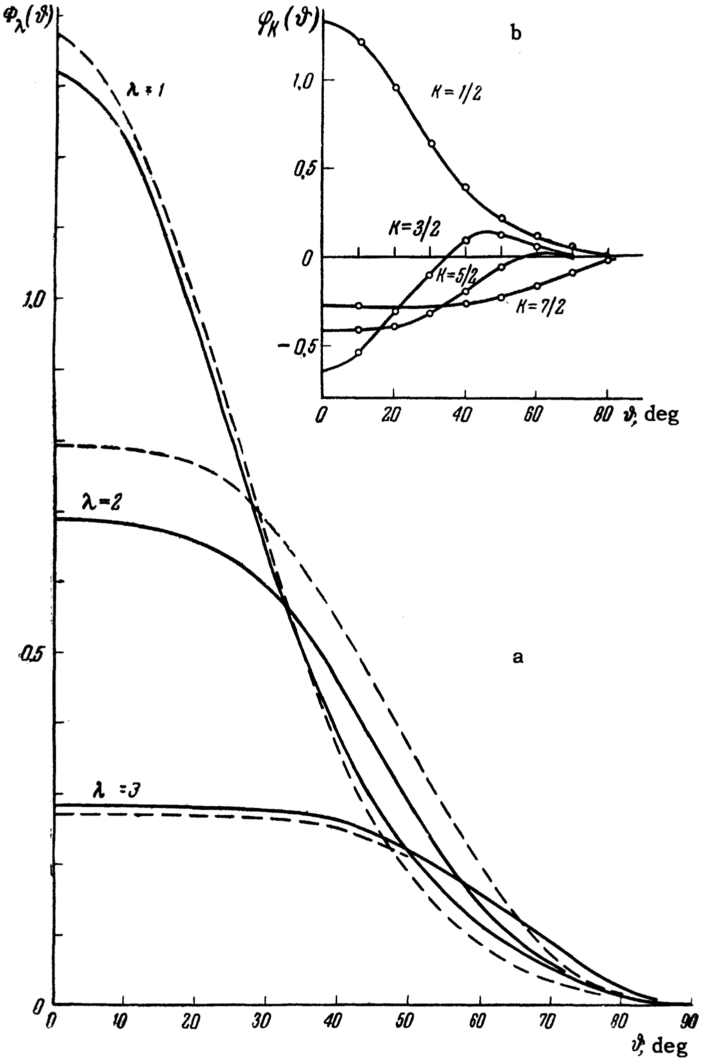

FIG. 1. a – functions $\Phi_\lambda(\vartheta)$ calculated for 1.5 Mev neutrons; solid curve corresponds to the optical model for a spherical nucleus with a diffuse boundary and with spin-orbit interaction (absorption coefficients are given in the text), dotted curve corresponds to the functions Φ_λ for a black nucleus and $l_{max} = 3$; b – functions $\varphi_K(\vartheta)$ calculated with formula (45) from the functions $\Phi_\lambda(\vartheta)$ for a semi-transparent nucleus ($E_n = 1.5$ Mev).

*The author expresses his indebtedness to P. É. Nemirovskiĭ for making available the results of his calculations of the absorption coefficients.

*For the determination of the value of K_0^2 from the experimental data, see references 8 and 9.

data of Blumberg and Leachman[8] for Pu^{239} and U^{233} apparently indicate the presence of substantial fluctuations in the distribution of a(K). Thus, on the basis of the angular distributions for $E_n = 1.5$, it has to be conceded that a(0) considerably exceeds the statistical value (it is 1.5 — 3 times as great).

The second term in formula (35) is associated with the selection rule for J in the state K = 0 for $J_0 = 5/2$ and $l_{max} = 3$-4, and does not exceed the value 0.02, which is within the limits of experimental error. For Pu^{239} ($J_0 = 1/2$), the amplitude of the second term is equal approximately to 0.15 — 0.20, i.e., larger than all values of the observed anisotropy. For the same values of the parameter $l^2/4K_0^2$ as for U^{233}, the presence of the second term in formula (35) should lead to the appearance of a relatively deep minimum at $\vartheta = 0°$ and a broad maximum at $\vartheta \approx 40°$, which is clearly contradicted by the experimental data.

Hence one may conclude that the interdiction on odd (or even) values of spin for an even (or odd) state of the nucleus does not occur, at least, for transition nucleus excitation energies $\gtrsim 3$ Mev. The absence of such an interdiction, well-known from weakly excited states of even-even deformed nuclei, may be associated with fluctuation deviations of the self-consistent field from axial symmetry. Such deviations may prove to be very important, since the criteria that the collective motion during fission be adiabatic are satisfied with only a small reserve[10] (see also reference 11). This would signify the impossibility of describing the nuclear state by means of a wave function of the rotational type. The quantum number K would have the meaning of a mean statistical constant of motion.

Comparison of the value of the anisotropy for U^{235}, U^{233}, and Pu^{239} (reference 9) indicates that there is a systematic tendency towards a small increase in the anisotropy for nuclei with larger values of J_0; the difference in the ratios $\sigma_f(0°)/\sigma_f(90°)$ is, in each case, approximately 0.03. This effect may be associated with the presence of the negative term proportional to K^4 in the expansion of a(K) in a series in K. Writing a(K) in the form

$$a(K) \approx 1 - (K^2/2K_0^2) - \varkappa K^4, \tag{40}$$

we obtain for the angular distribution

$$W(\vartheta) \approx 1 + (\overline{l^2}/4K_0^2)\cos^2\vartheta - \tfrac{3}{8}\varkappa\overline{l^4}\sin^4\vartheta + \varkappa(J_0 + 1/2)^2\,\overline{l^2}\cos^2\vartheta. \tag{41}$$

Since the first three terms in formula (41) do not depend on J_0, comparison of the values of the anisotropy for different values of J_0 permits one to determine the parameter κ at once. For the indicated difference in the ratios $\sigma_f(0°)/\sigma_f(90°)$, we obtain $\kappa = (2-4) \times 10^{-4}$. For such κ, the last two terms in formula (41) have practically no influence on the shape of the angular distribution. The presence in the expansion of a(K) of a negative term proportional to K^4 gives a relative decrease in the contribution of large K in comparison with a Gaussian distribution.

6. In the case of neutron-induced fission of even-even nuclei ($J_0 = 0$), the angular distribution of the fragments is also described by formulas (30) — (32), where the coefficients β_λ are given by

$$\beta_\lambda = [a(1/2 - \lambda) - a(1/2 + \lambda)]/2a(1/2) \tag{42}$$

(for $S = 1/2$, σ_f is different from zero only for $K = 1/2$). If, now, the fission is characterized mainly by one value $K = K^*$, the coefficients a(K) in the numerator of formula (42) can be represented in the form

$$a(K) \approx a(K^*)\{\delta_{K,K^*} + \delta_{K,-K^*}\},$$

from which we obtain for the function $A(\vartheta)$

$$A(\vartheta) = [a(K^*)/2a(1/2)]\,\varphi_{K^*}(\vartheta), \tag{43}$$

$$\varphi_{K^*}(\vartheta) = \begin{cases} \Phi_1(\vartheta), & K^* = 1/2 \\ -\Phi_{K^*-1/2}(\vartheta), & K^* = l_{max} + 1/2 \\ \Phi_{K^*+1/2}(\vartheta) - \Phi_{K^*-1/2}(\vartheta), & K^* \neq 1/2,\ l_{max} + 1/2. \end{cases} \tag{43'}$$

The functions $\varphi_{K^*}(\vartheta)$ can also be represented in the form

$$\varphi_{K^*}(\vartheta) = 4\pi \sum_{l \geqslant K^*-1/2} \zeta_l [\,|Y_{l,K^*-1/2}(\vartheta)|^2 + |Y_{l,K^*+1/2}(\vartheta)|^2] \Big/ \sum_l (2l+1)\zeta_l. \tag{43''}$$

For $K^* = 1/2$, we find

$$\sigma_f(\vartheta)/\sigma_f(90°) = 1 + \tfrac{1}{2}[\sigma_f(0°)/\sigma_f(90°)]\,\Phi_1(\vartheta), \tag{44}$$

where

$$\sigma_f(0°)/\sigma_f(90°) = \{1 - \tfrac{1}{2}\Phi_1(\vartheta)\}^{-1}.$$

The angular distribution for $K^* = 1/2$ is characterized by a sharp maximum for $\vartheta = 0°$ (the half-width of the maximum is of the order l_{max}^{-1}). For other values of K^*, the function $\sigma_f(\vartheta)$ has a minimum at $\vartheta = 0°$. The width of the minimum increases, and its depth decreases, for larger K^*. For small K^* not equal to $1/2$, there is also a small maximum at $\vartheta = 40 - 50°$. Figure 1 shows several functions $\varphi_K(\vartheta)$ calculated for a semi-transparent nucleus with $E_n = 1.5$ Mev. For another neutron energy (or other values of the absorption coefficients) the functions $\varphi_K(\vartheta)$, as is the case for the functions $\Phi_\lambda(\vartheta)$, can readily be calculated by formulas (32) — (43). For $E_n \lesssim 2$ Mev, the table can also be used.

For $K^* = 1/2$, the angular distribution of the fragments is determined single-valuedly if the function $\Phi_1(\vartheta)$ is known. For a neutron energy of 0.7 — 1.0 Mev, the value of $\Phi_1(0°)$ is 0.9 — 1.2, depending on the absorption coefficients. For $E_n = 0.7$ Mev, the optical model gives $\Phi_1(0°) = 1.05$ ($\overline{\zeta}_0 = 0.35$, $\overline{\zeta}_1 = 0.81$, $\overline{\zeta}_2 = 0.12$, $\overline{\zeta}_3 = 0.07$), from which, by formula (44), we find $\sigma_f(0°)/\sigma_f(90°) \simeq 2$. This value is close to the experimental value for Th^{230} (reference 12). Figure 2 shows the experimental data of Henkel and Brolley,[13] for fission of Th^{232} induced by 1.6-Mev neutrons. The solid and dotted curves represent the angular distributions calculated from formula (43) by means of the functions Φ_λ shown in Fig. 1 for a semi-transparent and a black nucleus, respectively, with $K^* = 3/2$. For the ratio of the coefficients, we have $a(3/2)/a(1/2) \approx 6 - 8$. The curve calculated with $\overline{\zeta}_l$ for the optical model is in good agreement with the experimental data.

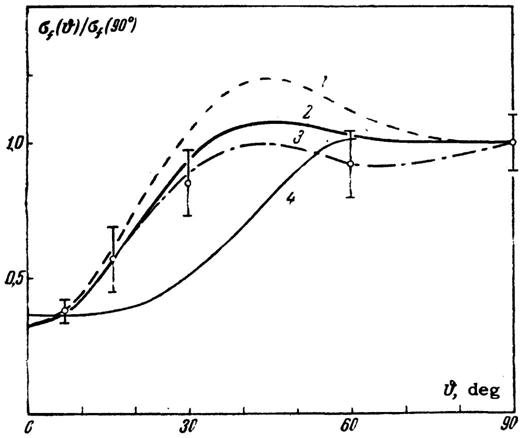

FIG. 2. Fragment angular distribution for fission of Th^{232} induced by 1.6-Mev neutrons, according to the data of Henkel and Brolley,[13] (experimental points). Curves 1 and 2 are the angular distributions for K = 3/2 for a semi-transparent and a black nucleus, respectively; Curve 3 is the "best" angular distribution for K = 3/2 determined by Wilets and Chase[14] by the method of least squares; Curve 4 was calculated from the optical model for K = 5/2.

The angular distribution of the fission fragments of Th^{232} was also analyzed by Wilets and Chase.[14] They represented the angular distribution in the form

$$\sum_{J=3/2}^{7/2} r(J)\,|D^J_{1/2\,3/2}(\vartheta)|^2 + \text{const},$$

where the coefficients r(J) were determined from the experimental points by the method of least squares. The curve obtained in this way is also shown in Fig. 2 (dash-dots). This curve differs little from the angular distribution calculated with the absorption coefficients for a semi-transparent nucleus. The difference between the solid and dotted curves in Fig. 2 is connected with the difference in the size of the contribution of $l = 2$ in the cases of black and semi-transparent nuclei. We note that in the case of a semi-transparent nucleus the choice of the parity of l, connected with the fact that the parity of the rotational states with respect to the same region should be the same, is not important, owing to the relatively small value of $\overline{\zeta}_2$ in comparison with $\overline{\zeta}_1$ and $\overline{\zeta}_3$ in the optical model.

As shown by Henkel and Simmons,[9] there are many other cases of an "anomalous" angular distribution of fission fragments of even-even nuclei at neutron energies of the order 0.5 — 1.5 Mev. These distributions have the shape of the curves shown in Fig. 1b and may apparently be explained by the anomalously large contribution of fission with some particular value of K. Thus, for example, the angular distribution of the fission fragments of U^{236} for $E_n = 0.85$ Mev[9] is similar to the curve corresponding to $K^* = 3/2$ in Fig. 1b. In principle, the most complete information on the distribution of a(K) can be obtained from the experimental data directly from formula (30).

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