PION-NUCLEON SCATTERING AT LOW ENERGIES, I

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Integral equations for πN-scattering partial waves in the low-energy region are derived on the basis of the Mandelstam representation. The kernels of the equations contain the lowest $\pi\pi$- scattering phases.

1. INTRODUCTION

THE Mandelstam double dispersion representation[1] for a two-particle Green's function makes it possible to consider the matrix elements of the interaction processes corresponding to this Green's function as different limiting values of the same analytic function of two complex variables. This representation leads to simple dispersion relations in one variable, including the usual dispersion relations in the energy for arbitrary values of momentum transfer. Using the unitarity condition for the imaginary parts of the different amplitudes, we arrive at the possibility of obtaining a system of relations between the amplitudes of the various processes.

The unitasity condition, which utilizes the expansion in a complete system of intermediate states, introduces an infinite set of corresponding amplitudes. Confining ourselves in the unitarity condition to the lowest two-particle mass state, we arrive at a system of equations for the matrix elements of two-particle processes. The two-particle approximation does not lead to an error in the imaginary parts up to the threshold of the next mass state, and consequently the integrated contribution of the latter can be considered small in the low-energy region. This approximation can be considered as formulated by Chew's statement that "the behavior of the analytic function in a small region is determined essentially by the nearest singularities."[2]

The program developed here yields equations for two-particle amplitudes in the low-energy region. It is clear that an important role is played here by the scattering amplitudes of the lightest particles. Neglecting electromagnetic effects, we obtain for "ordinary" strongly-interacting particles the sequence of processes shown in Fig. 1.

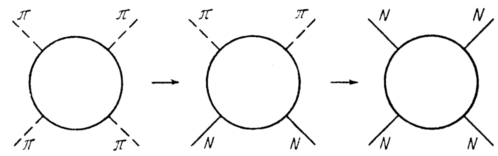

FIG. 1.

The scheme of Fig. 1 denotes, for example, that the system of equations for the processes πN $\to$ πN and $\pi\pi \to N\bar{N}$ contains the amplitudes of the process $\pi\pi \to \pi\pi$. For $\pi\pi$ scattering, to the contrary, the system of equations will be closed. Thus, the process $\pi\pi \to \pi\pi$ becomes very important in the theory of strong interactions, and serves as its "starting point."

The equations for the $\pi\pi$-scattering process were obtained by Chew and Mandelstam.[3] These equations, like the Chew-Low equations, are singular nonlinear integral equations. We know[4] that such equations have sets of solutions. Even the determination of some definite branch of these solutions is a complicated task, which can be accomplished only with the aid of high speed electronic computers. At the present time, a particular "adiabatic" solution of the Chew-Mandelstam equations has been obtained,[5] with the s wave predominating.

The block $\pi\pi$NN of Fig. 1, which describes the processes

$$\text{I.} \quad \pi + N \to \pi' + N,$$
$$\text{II.} \quad \pi' + N \to \pi + N,$$
$$\text{III.} \quad \pi + \pi' \to N + \overline{N},$$

was considered by Frazer and Fulco[6] as applied to the reaction III. They obtained for the amplitudes of process III equations, that contain the πN-scattering amplitudes and the $\pi\pi$-scattering phase shifts.

Attempts to investigate the $\pi\pi$NN block as applied to πN scattering were undertaken by McDowell.[7] He considered the analytic properties of the partial amplitudes of meson-nucleon scattering* in the complex plane of the variable s — the square of the total energy of the meson + nucleon system [see (3.1) below] and established the kinematic complex singularities due to the inequality of the masses. These singularities make it difficult to obtain the integral equations. Recently, however, Chou Hung-Yüan (private communication) was able to carry the corresponding derivations through to conclusion and obtained a system of integral equations for the partial-scattering waves and for the process $\pi\pi \rightarrow N\bar{N}$.

In the present paper we investigate the analytical properties of the πN scattering amplitudes in the complex plane of the square of the momentum, variable q^2, in the c.m. system for a fixed scattering angle. This approach is analogous to that of Cini-Fubini-Stanghellini[8] to the $NN \rightarrow NN$ process.

In the plane of the variable q^2, the kinematic singularities have the form of a finite cut on the real axis and can be eliminated by considering the corresponding symmetrized and antisymmetrized expressions.

Confining ourselves in the unitarity conditions for reaction III to the two-meson state, we leave in the $\pi\pi \rightarrow \pi\pi$ amplitudes only the s and p phase shifts. This allows us to obtain for the amplitudes of processes I and II an integral equation that contains only the aforementioned two phase shifts δ_0 and δ_1 for $\pi\pi$ scattering. The use of the cited Chew principle allows us to neglect the non-physical cuts due to reaction II and to obtain integral equations for the partial waves of πN scattering. Naturally, these equations can have a meaning only in the region of low energies.

Section 2 contains a summary of the formulas for the matrix elements of processes I, II, and III in terms of the invariant coefficients of the two-particle Green's function, and also the unitarity conditions for the partial waves.

In Sec. 3 we choose the functions and the variable for the analytic continuation. This continuation, as well as the elimination of the kinematic cuts, is realized in Sec. 4. Next (Sec. 5) we analyze the nearest part of the non-physical cut of reaction III, by means of an approximate unitarity condition which contains only the s and p phase shifts of the $\pi\pi$ scattering. With the aid of the Muskhelishvili method we obtain for the amplitudes of πN scattering integral equations that contain integrals only over the physical regions of reactions I and II. The $\pi\pi$-scattering phase shifts enter into the kernels of these equations. In Sec. 6 we go over to the partial waves in the final equations.

*McDowell actually considered the process $K + N \rightarrow K + N$, but the kinematic singularities of this process are analogous to those of the process $\pi + N \rightarrow \pi + N$.

The foregoing brief survey of papers on the use of the Mandelstam representation is incomplete. Mention should be made of the very important paper by Ter-Martirosyan,[9] in which the unitarity conditions in the two-particle approximation, subject to an imposed crossing symmetry, are used to obtain equations for the spectral functions of the Mandelstam representation. An interesting method for the approximate reduction of the double Mandelstam representation to a sum of one-dimensional representations was proposed by Cini and Fubini.[10] Their technique makes it possible to simplify the derivation of the equations for the partial waves.

2. MATRIX ELEMENTS, UNITARITY CONDITIONS

The matrix elements of processes I, II, and III, described by the block $\pi\pi NN$, are represented in the form

$$\langle f|S-1|i\rangle = \frac{i}{4\pi^2}\delta(p_1+p_2+q_1+q_2)\sqrt{\frac{M^2}{4p_1^0p_2^0q_1^0q_2^0}}\,\bar{u}Tu, \qquad (2.1)$$

p_1 and p_2 are the 4-momenta of the nuclei while q_1 and q_2 are those of the mesons. The two-particle Green's function T has the following structure:

$$T = A + \tfrac{1}{2}(\hat{q}_1 - \hat{q}_2)B, \qquad T = \delta_{\rho\rho'}T^{(+)} + \tfrac{1}{2}[\tau_\rho, \tau_{\rho'}]T^{(-)} \qquad (2.2)$$

The notation in (2.1) and (2.2) is standard; the Feynman metric is used: $\hat{q} = q^0\gamma_0 - \gamma\mathbf{q}$, $\gamma_0^2 = 1$, and $\gamma_\alpha^2 = -1$.

The matrix elements $\bar{u}Tu$ coincide, accurate to within a factor $M/4\pi W$ (W is the total energy of the process), with the 'spiral states' of Jacob and Wick. For the scattering process, these states have the form

$$f_{++} = f_{--} = \cos\frac{\theta}{2}(f_1+f_2) = \cos\frac{\theta}{2}\,\frac{MA + (p^0W - M^2)B}{4\pi W},$$

$$f_{+-} = -f_{-+} = \sin\frac{\theta}{2}(f_1 - f_2) = \sin\frac{\theta}{2}\,\frac{p^0A + Mq^0B}{4\pi W}. \qquad (2.3)$$

Here, as in (2.6) below, the azimuth angles are assumed equal to zero; θ is the scattering angle in the c.m.s. and f_1 and f_2 are expressed in terms of the partial amplitudes:

$$f_1 = \sum_l \{f_{l,+}P'_{l+1}(\cos\theta) - f_{l,-}P'_{l-1}(\cos\theta)\},$$

$$f_2 = \sum_l (f_{l,-} - f_{l,+})P'_l(\cos\theta), \qquad (2.4)$$

$$f^J_{l,\pm} = q^{-1}\sin\delta^J_{l,\pm}\exp(i\delta^J_{l,\pm}). \qquad (2.5)$$

The spiral states for the process III will be (cf. reference 6)

$$J_{++}=J_{--}=\frac{i}{8\pi}\left\{-\frac{pA}{p^0}+\frac{qM\cos\theta_3}{p^0}B\right\},$$

$$J_{+-}=-J_{-+}=\frac{q\sin\theta_3}{8\pi}B. \qquad (2.6)$$

Here θ_3 is the angle between the vectors **p** and **q**. The expansion in the partial waves has the form

$$J_{++}=\frac{1}{pp^0}\sum_l\left(l+\frac{1}{2}\right)(pq)^l f_+^l P_l(\cos\theta_3), \qquad (2.7)$$

$$J_{+-}=q\sum_l\frac{l+1/2}{\sqrt{l(l+1)}}(pq)^{l-1}f_-^l P_l^{(1)}(\cos\theta_3). \qquad (2.8)$$

If we introduce into these formulas the isotopic indices + and −, then the summation goes only over even l for the + sign and over odd l for the − sign.

We consider the antihermitian part of the third process, confining ourselves to the two-meson term in the sum over the complete system of functions. This system can be represented in the form

$$\operatorname{Im}J_{\lambda\mu}^{(\pm)}=\frac{q}{64\pi^2q^0}\int d\Omega_{\mathbf{q}'}J_{\lambda\mu}^{(\pm)}(\mathbf{p},-\mathbf{p};\ \mathbf{q}',$$

$$-\mathbf{q}')\,\Pi^{(0,1)*}(\mathbf{q}',\ -\mathbf{q}';\ \mathbf{q},-\mathbf{q}). \qquad (2.9)$$

Here Π^0 and Π^1 are the $\pi\pi$ scattering amplitudes with total isotopic spin 0 and 1, respectively [see, for example, formula (2.8) in the paper by Chew and Mandelstam[3]]. It is seen from (2.9) that $J^{(\pm)}$ are the amplitudes of process III, with total isotopic spins 0 and 1.

We shall consider below the analytic continuation of (2.9) in the non-physical region of small q^2. In this case we can confine ourselves to the s and p waves in the amplitude Π. Expanding (2.9) in partial waves and expressing the results through A and B, we obtain in this approximation

$$\operatorname{Im}A^{(+)}=-4\pi p^{-2}f_+^{0(+)}e^{-i\delta_0}\sin\delta_0,$$

$$\operatorname{Im}B^{(-)}=6\sqrt{2}\,\pi f_-^{1(-)}e^{-i\delta_1}\sin\delta_1.$$

$$\operatorname{Im}A^{(-)}=12\pi qp^{-1}\cos\theta_3\{2^{-1/2}f_-^{1(-)}-f_+^{1(-)}\}e^{-i\delta_1}\sin\delta_1,$$

$$\operatorname{Im}B^{(+)}=0. \qquad (2.10)$$

Here δ_0 is the s phase with J = 0 and δ_1 the p phase for J = 1.

3. KINEMATICS; CHOICE OF VARIABLES AND FUNCTIONS FOR THE ANALYTIC CONTINUATION

We introduce the ordinary invariants

$$s=(p_1+q_1)^2,\ \ \bar{s}=(p_1+q_2)^2,\ \ t=(p_1+p_2)^2,$$

$$s+\bar{s}+t=2(\mu^2+M^2), \qquad (3.1)$$

which have the following form in c.m.s. of the reactions I and III

$$s=M^2+\mu^2+2q^2+2\sqrt{q^2+\mu^2}\sqrt{q^2+M^2},$$

$$\text{I}\quad \bar{s}=M^2+\mu^2-2q^2\cos\theta-2\sqrt{q^2+\mu^2}\sqrt{q^2+M^2}, \qquad (3.2)$$

$$t=-2q^2(1-\cos\theta);$$

$$s=-2q_3^2+M^2-\mu^2+2p_3q_3\cos\theta_3,$$

$$\bar{s}=-2q_3^2+M^2-\mu^2-2p_3q_3\cos\theta_3,$$

$$\text{III}\quad t=4(q_3^2+\mu^2)=4(p_3^2+M^2). \qquad (3.3)$$

The analytic properties of the invariant scalar functions $A^{(\pm)}$ and $B^{(\pm)}$, introduced in (2.2), are described by the Mandelstam representation[1] [see formula (2.12) of that paper]. It is important for what is to follow that the representations for $B^{(\pm)}$ contain the pole terms

$$B^{(\pm)}(s,\bar{s},t)=\frac{g^2}{M^2-s}\pm\frac{g^2}{M^2-\bar{s}}+\cdots, \qquad (3.4)$$

while the representations for $A^{(\pm)}$ do not contain these terms.

We choose for the analytic continuation the four functions Φ:

$$\Phi(s,\bar{s},t)=A^{(+)},\ \alpha,\ \beta,\ B^{(-)};$$

$$\alpha=A^{(-)}/(s-\bar{s}),\qquad \beta=B^{(+)}/(s-\bar{s}). \qquad (3.5)$$

All these functions have the property

$$\Phi(s,\bar{s},t)=\Phi(\bar{s},s,t). \qquad (3.6)$$

In addition, they diminish at infinity not slower than A and B. Division by $s-\bar{s}$ does not introduce any new singularities.

We shall consider the functions Φ as applied to reaction I. We fix the scattering angle as $\cos\theta = c$, i.e., we consider the analytical properties in the variable q^2. It is now convenient to rewrite (3.2) in the form

$$s=s(q^2)=R(q^2)+D(q^2)+2K(q^2),$$

$$\bar{s}=\bar{s}(q^2)=R(q^2)-D(q^2)-2K(q^2),$$

$$t=-2q^2(1-c), \qquad (3.7)$$

where

$$R(q^2)=M^2+\mu^2+q^2(1-c),\ D(q^2)=q^2(1+c),$$

$$K(q^2)=\sqrt{(q^2+M^2)(q^2+\mu^2)}. \qquad (3.8)$$

4. ELIMINATION OF THE KINEMATIC CUTS AND THE CAUCHY THEOREM

The functions $\Phi(s,\bar{s},t)=\Phi(q^2)$ for $\cos\theta = c$), considered in the complex plane of q^2, have the following singularities: 1) a cut $-\infty < q^2 < -2\mu^2/(1-c) = -a_3$, connected with the denominator $t' - t$ in the spectral formula of Mandelstam; 2) a cut $0 < q^2 < \infty$ due to the denominator $s' - s$; 3) a cut $-\infty < q^2 < -a(c)$ due to $\bar{s}' - \bar{s}$

$$a(c) = \begin{cases} M^2 & 1 \geqslant c \geqslant \mu/M \\ a_2 = (M^2 + \mu^2 - 2M\mu c)(1-c^2)^{-1}, & \mu/M \geqslant c \geqslant -1; \end{cases} \tag{4.1}$$

4) a cut $-M^2 < q^2 < -\mu^2$, due to the square root (3.8) in the dependence of s and $\bar{s}$ on q^2. We call this the kinematic cut.

In addition, the functions β and $B^{(-)}$ have poles connected with (3.4).

On changing to the variable s, the kinematic cut $-M^2 < q^2 < -\mu^2$ gives a complex cut lying on a circle (see reference 7). To get rid of this cut, we represent Φ in the following manner

$$\Phi(q^2, K) = \Phi_1(q^2) + K(q^2)\Phi_2(q^2), \tag{4.2}$$

distinctly separating the irrational dependence. The functions Φ_1 contain no irrational dependences. They can be determined as

$$\Phi_1(q^2) = \frac{1}{2}[\Phi(q^2, K) + \Phi(q^2, -K)], \tag{4.3}$$

$$\Phi_2(q^2) = \frac{1}{2}[\Phi(q^2, K) - \Phi(q^2, -K)]K^{-1}(q^2). \tag{4.4}$$

Let us explain the meaning of the function $\Phi(q^2, -K(q^2))$. According to (3.7), we have $\Phi(q^2, -K) = \Phi(s^*, \bar{s}^*, t)$, with

$$s^*(q^2) = R(q^2) + D(q^2) - 2K(q^2), \quad \bar{s}^*(q^2) = R(q^2) - D(q^2) + 2K(q^2). \tag{4.5}$$

The variables s^*, and $\bar{s}^*$ are given here in terms of the variables q^2 and c of reaction I.

The point $s^*, \bar{s}^*, t$ lies in the physical region of reaction II if (q^2, c) lies in the physical region of reaction I. The connection between these points is given by

$$q_2^2 = q^2 \frac{M^2 + \mu^2 + q^2(1+c^2) - 2cK(q^2)}{M^2 + \mu^2 - 2q^2c + 2K(q^2)},$$

$$c_2 = 1 - \frac{q^2}{q_2^2}(1-c). \tag{4.6}$$

[for a geometrical interpretation of relations (4.6) see the appendix].

The functions $\Phi(q^2, -K)$, considered in the complex plane of q^2, have the following singularities: 1) a cut $0 < q^2 < \infty$ due to the denominator $\bar{s}' - \bar{s}^*$ in the spectral formula of Mandelstam; 2) a cut $-a_2 < q^2 < -M^2$ due to the same denominator when $c \geq \mu/M$; 3) a cut $-\infty < q^2 < -2\mu^2/(1-c)$ due to the denominator $t' - t$; 4) a kinematic cut $-M^2 < q^2 < -\mu^2$. In addition, the functions β and $B^{(-)}$ of the arguments q^2 and $K(q^2)$ have poles.

The functions $\Phi_1(q^2)$ have all the singularities of the functions $\Phi(q^2, K)$ and $\Phi(q^2, -K)$, with the exception of the kinematic cut.

Writing the Cauchy formulas for these and returning to the initial function $\Phi(q^2, K)$, we obtain

$$\begin{aligned} \pi\Phi(q^2, K) = & \int_0^\infty [\operatorname{Im}\Phi(q'^2, K(q'^2))\varkappa_+(q^2, q'^2) \\ & + \operatorname{Im}\Phi(q'^2, -K')\varkappa_-(q^2, q'^2)] \frac{dq'^2}{q'^2 - q^2} \\ & + \int_{-\infty}^{-a_3} [\operatorname{Im}\Phi(q'^2, K')\varkappa_+(q^2, q'^2) \\ & + \operatorname{Im}\Phi(q'^2, -K')\varkappa_-(q^2, q'^2)] \frac{dq'^2}{q'^2 - q^2} \\ & + \int_{-\infty}^{-a(c)} \frac{\operatorname{Im}\Phi(q'^2, K')\varkappa_+(q^2, q'^2)}{q'^2 - q^2} dq'^2 \\ & + \theta\left(c - \frac{\mu}{M}\right) \int_{-a_2}^{-M^2} \frac{\operatorname{Im}\Phi(q'^2, -K')\varkappa_-(q^2, q'^2)}{q'^2 - q^2} dq'^2 + \text{pole} \end{aligned}$$

terms for β and $B^{(-)}$. (4.7)

Here

$$\varkappa_\pm(q^2, q'^2) = \tfrac{1}{2}(1 \pm K(q^2)/K(q'^2)). \tag{4.8}$$

Figure 2 shows the integration domains in (4.7) as functions of c. It is seen from this figure that the cases of backward and forward scattering are the simplest. Thus, when $c = -1$ the non-physical cut due to reaction (2) (domain II′) goes to infinity, only the cut due to reaction III remains at $q^2 < 0$. It can be shown that in this case the integral contains the antihermitian part of the amplitude of the physical value $\cos\theta_3 = -1$.

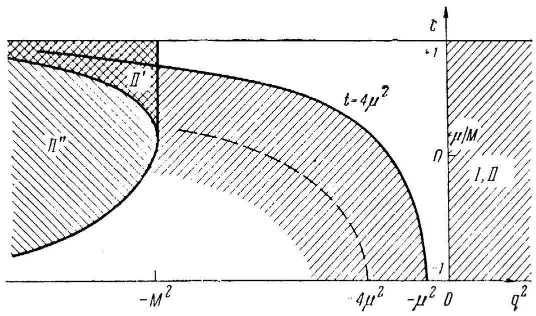

FIG. 2

For forward scattering $(c = +1)$, to the contrary, it is the cut due to reaction III that goes to infinity. In this case the antihermitian parts under the integrals over the domains II′ and II″ also depend on the physical value $\cos\theta_2 = +1$. These relations coincide exactly with usual forward dispersion relations in the energy variable E in the laboratory system of coordinates. The integration domains are as listed on the table.

	Domain of variable q^2	Domain of variable E
I	$0, \infty$	μ, ∞
II	$0, \infty$	$-(M^2+\mu^2)/2M, -\mu$
II′	$-\infty, -M^2$	$-M, -(M^2+\mu^2)/2M$
II″	$-\infty, -M^2$	$-\infty, -M$

5. INVESTIGATION OF THE CUT DUE TO REACTION III

The second integral (4.7) contains the antihermitian part of the amplitude of reaction III in the non-physical region $t < 4M^2$. We shall consider the quantities Im Φ in this region as analytic continuations of the corresponding functions from the physical region $t > 4M^2$. Their explicit expressions can be obtained from (2.2) by recalling that $s - \bar{s} = 4p_3q_3 \cos\theta_3$.

In view of the fact that in our approximation (only the s and p phase shifts of the $\pi\pi$ scattering are considered) the expressions (2.10) do not contain $\cos\theta_3$, they coincide with the numerator in the second integral of (4.7). In the more general case, when the higher-order phase shifts are considered, these expressions depend on $\cos\theta_3$ and the following connection must be used

$$2(\mathbf{p}_3\mathbf{q}_3) = \tfrac{1}{2}(s - \bar{s}) = D(q^2) + 2K(q^2). \qquad (5.1)$$

It is seen from (5.1) that $\cos\theta_3$ becomes complex, but the numerator of the second integral in (4.7) remains real. In order to connect Im Φ with the functions Φ themselves, it is necessary to turn to expansions of the type (2.7) and (2.8).

These expansions are really in the parameter $s - \bar{s}$. The radius of convergence of such an expansion is determined by the singularities of the Mandelstam representation. The nearest singularities are the poles due to the single-nucleon denominators. In the approximation (2.10) these singularities are essential only for the function $B^{(-)}$. We shall consider them later.

The following singularities are due to the integrals of the double spectral representation. It can be shown that these singularities do not prevent expansion up to

$$q'^2 \sim -2.3\ \mu^2. \qquad (5.2)$$

Therefore the expansions (2.7) and (2.8) are correct only in the region (5.2). We assume that the terms of these expansions diminish rapidly and confine ourselves to the first terms. This yields

$$\operatorname{Im} A^{(+)} = A^{(+)} e^{-i\delta_0} \sin\delta_0, \qquad \operatorname{Im}\alpha = \alpha e^{-i\delta_1} \sin\delta_1. \qquad (5.3)$$

Let us turn to Eq. (4.7) and let us compare the effect due to different non-physical contributions from the interval $q'^2 < 0$. Assuming henceforth a transition to the partial scattering amplitudes, i.e., averaging over $\cos\theta = c$, we should examine the roles of these cuts in the variable q^2 independently of c. It follows from Fig. 2 that the nearest cut, beginning with the point $-\mu^2$, is connected with the reaction III. For arbitrary c, it is bounded by the curve $t = 4\mu^2$.

The use of the unitarity conditions with two mesons in the intermediate state allows us to take accurate account of this cut up to the point $-4\mu^2$ (there are grounds[3,6,12] for assuming that these relations will be true also in a certain region below $-4\mu^2$). However, as noted earlier [see (5.2)], formulas (5.3) cannot be correct for values lower than $-2.3\mu^2$. Thus, it is sensible to expect that formulas (2.3) will enable us to take into account the nearest non-physical singularities in the interval $-2.3\,\mu^2 < q'^2 \leq -\mu^2$.

It is clear that we should therefore neglect completely the contributions due to the nonphysical regions II′ and II″, located behind $-M^2$. The estimates for the forward scattering, when regions II′ and II″ make up the entire non-physical contribution, have shown that their neglect at small energies, down to values of E on the order of 100 Mev, causes an error of approximately 10%. The global effect is expected to decrease after averaging over the angle. As a result we obtain the following equation for β:

$$\beta(q^2, K) = \frac{g^2}{(M^2 - s)(M^2 - \bar{s})} + \frac{1}{\pi}\int_0^\infty \operatorname{Im}\beta(q'^2, K') \frac{\varkappa_+(q^2, q'^2)\, dq'^2}{q'^2 - q^2} + \frac{1}{\pi}\int_0^\infty \operatorname{Im}\beta(q'^2, -K') \frac{\varkappa_-(q^2, q'^2)}{q'^2 - q^2} dq'^2. \qquad (5.4)$$

The corresponding equations for $A^{(+)}$ and α can be simplified by returning to the functions Φ_1 and Φ_2. In our approximation we have Im $\Phi_2 = 0$ and Im $\Phi_1 = \Phi_1 e^{-i\delta} \sin\delta$ on the cut $(-2.3\mu^2, -a_3)$. If we consider the integrals over the domain $0, \infty$ to be known functions, then the equation for Φ_1 will be a linear singular equation, which can be solved by Muskhelishvili's method (see Chapter 5 of the book by Muskhelishvili,[13] and also the paper by Omnes[14]). In our case this method reduces to an examination not of $\Phi_1(q^2)$ but of the function $\Phi_1(q^2)\exp[-u(q^2)]$, where

$$u(q^2) = -\frac{1}{\pi}\int_{-2,3\mu^2}^{-a_3} \frac{\delta(-q'^2(1-c)/2-\mu^2)}{q'^2-q^2}\,dq'^2, \qquad (5.5)$$

which has no singularities when $q^2 < 0$.

Solving the equation in this manner, we obtain after simple transformations

$$\Phi(q^2, K) = \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\Phi_{\rm I}(q'^2)\,\varkappa_+(\delta, q^2, q'^2)}{q'^2-q^2}\,dq'^2$$
$$+\frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\Phi_{\rm II}(q'^2)\,\varkappa_-(\delta, q^2, q'^2)}{q'^2-q^2}\,dq'^2, \quad \Phi_{\rm I,\,II} = \Phi(q^2, \pm K), \qquad (5.6)$$

$$\varkappa_\pm(\delta, q^2, q'^2) = \tfrac{1}{2}\,[\exp[u(q^2)-u(q'^2)] \pm K(q^2)/K(q'^2)]. \qquad (5.7)$$

We note that our approximation (5.3) is equivalent to the condition $\Phi_2(q^2) = 0$ when $q^2 < -a_3$.

Let us return to the function $B^{(-)}$, for which the Mandelstam representation contains pole terms which make impossible an expansion in the argument $s-\bar{s}$ in the domain of the reaction III. We shall assume that the large values of the real parts of the higher partial waves f^l ($l \geq 3$) in the expansion (2.8) are due entirely to these pole terms. (Analogous considerations were used by Okun' and Pomeranchuk[15] in an analogous analysis of the higher partial waves in NN scattering.) This enables us to write

$$\operatorname{Im} B^{(-)} = \{B^{(-)} - g^2(\Delta - \langle\Delta\rangle_1)\}\, e^{-i\delta_1}\sin\delta_1. \qquad (5.8)$$

Here Δ denotes the pole term, and $\langle\Delta\rangle_1$ its first term in the expansion of the angle of reaction III,

$$\langle\Delta\rangle_1 = \frac{1}{2}\int_{-1}^{+1}\frac{2M^2-s-\bar{s}}{(M^2-s)(M^2-\bar{s})}\,d\cos\theta_3$$
$$= \frac{1}{2Z(t)}\ln\frac{R(t)+2Z(t)}{R(t)-2Z(t)},$$
$$Z(t) = \sqrt{(t/4-\mu^2)(t/4-M^2)}, \qquad (5.9)$$

where we choose that branch of the function $\langle\Delta\rangle_1$ which is real in the physical region of reaction I.

Carrying out the transformation with e^{-u}, we obtain

$$B^{(-)} = g^2(\Delta - \langle\Delta\rangle_1)$$
$$+\frac{1}{\pi}\int_0^\infty \frac{\varkappa_+(\delta, q^2, q'^2)\operatorname{Im} B_{\rm I}^{(-)} + \varkappa_-(\delta, q^2, q'^2)\operatorname{Im} B_{\rm II}^{(-)}}{q'^2-q^2}\,dq'^2. \qquad (5.10)$$

In our approximation, the part of the function $B^{(-)}$ which is antisymmetrized in K is equal to the antisymmetrized part of the pole terms, i.e., $B_2^{(-)} = g^2\Delta_2$. We note further that an analogous method of eliminating the difficulties connected with the poles was recently proposed by Cini and Fubini.[10]

6. TRANSITION TO THE PARTIAL SCATTERING AMPLITUDES

In order to convert relations (5.4), (5.6), and (5.10) into a closed system of equations, it remains for us to use the unitarity relations and to go over to the partial scattering amplitudes. From (2.3), with allowance for the fact that

$$s - \bar{s} = 2(1+\cos\theta)\,q^2 + 4q^0p^0,$$

we obtain expression for the functions Φ in terms of the variables and amplitudes of the reaction I. The $f_{1,2}^{(\pm)}$ which are contained therein depend on the arguments q^2 and $\cos\theta = c$, and are expressed with the aid of (2.4) in terms of the partial waves f_l, which according to (2.5), have the unitarity property.

Thus, we can go over to the partial amplitudes f_l either from the function Φ or from $\operatorname{Im}\Phi_1(q'^2)$, which enter into the integral terms of the assembly (5.4), (5.6), and (5.10). In order to go over to f_l from $\operatorname{Im}\Phi_{\rm II} = \operatorname{Im}\Phi(q'^2, -K)$, we note, firstly, that formulas (2.3) retain their form also in the variables of the reaction II. For this purpose it is necessary only to go over in these formulas to the corresponding variables in the c.m.s. of the reaction II: $\cos\theta \to \cos\theta_2$, $q \to q_2$, $q^0 \to q_2^0$, and $p^0 \to p_2^0$. On the other hand, we transform to the variables of reaction II in the arguments $\operatorname{Im}\Phi_{\rm II}$ with the aid of formulas (4.6).

From these we obtain

$$\cos\theta_2 = [2q_0^2 + M^2 + \mu^2 - 2cK(q_2^2)$$
$$- \square(q_2^2, c)]/2q_2^2(1+c),$$
$$\square(q_2^2, c) = \{[2q_2^2 + 2K(q_2^2) - c(M^2+\mu^2)]^2$$
$$+(1-c^2)(M^2-\mu^2)^2\}^{1/2}. \qquad (6.1)$$

Going over to the new integration variable q'^2 in the integrals that contain $\operatorname{Im}\Phi_{\rm II}(q'^2)$,

$$q'^2 = [\square(q_2'^2, c) + 2c[q_2'^2 + K(q_2'^2)] - M^2 - \mu^2]/2(1-c^2),$$
$$K(q'^2) = [2K(q_2'^2) + 2q_2'^2 - c(M^2+\mu^2-\square)]/2(1-c^2),$$
$$\frac{dq'^2}{dq_2'^2} = \frac{2K(q_2'^2) + M^2 + \mu^2 - 2q^2}{\square(q_2'^2, c)}\,\frac{K(q'^2)}{K(q_2'^2)} = D(q_2'^2, c), \qquad (6.2)$$

we can represent these in the form

$$\int_0^\infty \frac{\operatorname{Im}\Phi(q'^2)}{q'^2-q^2}\,\varkappa_-(\delta, q^2, q'^2)\,dq'^2$$
$$= \int_0^\infty \frac{\operatorname{Im}\Phi(q_2'^2, c_2(q_2'^2, c))}{q'^2(q_2'^2, c)-q^2}\,D(q_2'^2, c)\,\varkappa_-(\delta, q_2'^2, c, q^2)\,dq_2'^2. \qquad (6.3)$$

Here $\Phi(q_2'^2, c_2)$ denotes the amplitude of reaction II, written down in terms of the momentum q_2' and the cosine of the scattering angle c_2 in its c.m.s. $\operatorname{Im}\Phi(q_2'^2, c_2)$ is determined from (2.3) — (2.5). The symbol $\kappa_-(\delta, q_2'^2, c, q^2)$ denotes the value of (5.7) after the substitution (6.2).

7. DISCUSSION OF THE RESULTS

Relations (5.4), (5.6), (5.10), and (6.1) — (6.3) form a complete system of equations for the partial waves of πN scattering. In the derivation we took accurate account of the pole terms and of the contribution of the $\pi\pi$ interaction up to $q'^2 = -2.3\,\mu^2$, where the four-meson contributions to the process $\pi\pi \to N\bar{N}$ and the contributions due to the Mandelstam double representations of the spectral functions become significant. The equations obtained can therefore give sensible approximations only in the region of small energies, where we can confine ourselves to a small number of partial waves.

The equations contain the phase shifts δ_0 and δ_1 for the $\pi\pi$ scattering. Since we have at present no direct information on these phase shifts, it is natural to use the obtained system of equations as an indirect source of information on these phase shifts. Of great interest here is the verification of the hypothesis[16] that resonance exists in the $\pi\pi$-scattering p wave. Such a hypothesis leads to sensible results with respect to electromagnetic structure of the nucleon.[17]

We note that the procedure outlined for eliminating the kinematic singularities can be used from any other problems in the scattering of particles with unequal masses, for example, for Kπ and KN scattering.

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APPENDIX*

Let us explain the geometrical meaning of the transformation (4.6). We describe the kinematics of the reactions in a Lobachevskiĭ-type velocity space.[18] The usual connection between the elements of the triangle $C^2 = A^2 + B^2 - 2AB\cos\gamma$ is replaced here by $\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos\gamma$. The 4-velocity p/m of the particle is represented by a point in this space.

We consider a plane in velocity space, passing through the velocities of the particles that participate in reaction I: $P_i = p_i/M$, $Q_i = q_i/\mu$ $(i = 1, 2)$ (see Fig. 3).

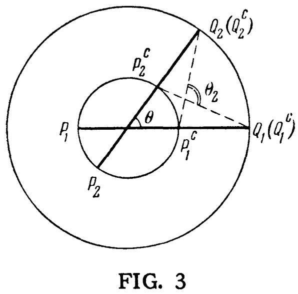

FIG. 3

The point 0 (intersection of the lines P_1Q_1 and P_2Q_2) represents the velocity of the center of mass. The invariants s and t can be expressed here in terms of the distances $\overline{P_1Q_1}$ and $\overline{Q_1Q_2}$:

$$s = M^2 + \mu^2 + 2M\mu\cosh\overline{P_1Q_1}, \qquad t = 2\mu^2(1 - \cosh\overline{Q_1Q_2}).$$

In order to go over to the variable of the second reaction, we note that it follows from (3.2) and (4.5) that $ss^* = (M^2 - \mu^2)^2$. It follows therefore that s^* can be represented in the form

$$s^* = M^2 + \mu^2 - 2M\mu\cosh\overline{P_1^cQ_1}.$$

If we now identify P_1^c with the velocity of the initial nucleon in reaction II, and Q_1 with the velocity of the escaping meson in the reaction II $-Q_1^c$, then it follows from the invariance of t that the point Q_2 can be identified with the velocity Q_2^c. Then P_2^c is the intersection of the segment $0Q_2$ with the small circle. The velocity $0'$ of the center of mass of the reaction II and the scattering angle θ_2 are determined by the intersection of the segment $P_2^cQ_1$ and $Q_2P_1^c$ and the angle between them. The momentum of reaction II is determined by the relation

$$q_2 = M\sinh\overline{O'P_1^c} = \mu\sinh\overline{O'Q_1^c}.$$

It is seen from the figure that $\theta_2 > \theta$, with the exception of the forward and backward scattering, when these two angles are equal, while $q_2 = q$ for backward scattering and is less than q for all other angles.

*The interpretation given here was proposed by N. A. Chernikov.

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