CHANGE OF THE ADIABATIC INVARIANT OF A PARTICLE IN A MAGNETIC FIELD. I

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The change of the adiabatic invariant is found for a particle moving in an axially symmetrical inhomogeneous magnetic field. The problem is solved for the usual model Hamiltonian (1).

THE problem of the conservation of adiabatic invariants, and in particular of the magnetic moment of a charged particle in a magnetic field, has recently been dealt with in a number of papers;[1,2] these papers have shown that the change of the adiabatic invariant is less than any power of the adiabatic parameter. In reference 3 the change of the adiabatic invariant was calculated for an oscillator whose frequency depends on the time. It is obvious that the change of the adiabatic invariant must be the same for a particle in a time-varying uniform magnetic field, under the condition $\mathbf{H} \rightarrow$ const as $t \rightarrow \pm\infty$. A more complicated problem is to calculate the change of the adiabatic invariant in a magnetic field that varies in space. The solution of this problem is the purpose of the present paper.

1. STATEMENT OF THE PROBLEM AND METHOD OF SOLUTION

Suppose the magnetic field varies slowly with position (the change in a distance equal to the Larmor radius being small). For $x \rightarrow \pm\infty$ we have $\mathbf{H} \rightarrow \mathbf{H}_{\pm}$. The particle coming in from $-\infty$ has the magnetic moment I_-. As $x \rightarrow +\infty$, the magnetic moment of the particle will approach some other value I_+. We shall calculate the change $I_+ - I_-$. The result can be obtained qualitatively by the following simple method.

As is well known (cf. e. g., reference 4), if the curvature of the lines of force is neglected the problem reduces to the study of the model Hamiltonian

$$H = (p_x^2 + p_y^2 + m^2\omega^2(x)\,y^2)/2m, \tag{1}$$

where $\omega(x)$ is the Larmor frequency. The small parameter of the problem is the quantity $\alpha = r_L\omega'/\omega$, where r_L is the Larmor radius. Suppose the solution $x = x(t)$ is known for the coordinate x. Substituting this function into the equation of motion for y, we arrive at the problem of the change of the adiabatic invariant of an oscillator with frequency depending on the time, which was solved in reference 3. The result was found to be

$$\frac{\Delta I}{I} = 2\mathrm{Re}\left\{-i\exp 2i\left[\int^{t_0}\omega(t)\,dt + \varphi_-\right]\right\}, \tag{2}$$

where t_0 is a zero of the function $\omega(t)$ in the complex plane of t, and φ_- is the phase of the oscillator at $t \rightarrow -\infty$.* Since the adiabatic invariant changes only slightly during the motion, we shall treat it as a constant in obtaining the time dependence $x(t)$. Setting I = const in the Hamiltonian (1) we get

$$dt = m\,dx/\sqrt{2m(E - I\omega(x))}. \tag{3}$$

Finally, substituting Eq. (3) in Eq. (2), we get

$$\frac{\Delta I}{I} = 2\mathrm{Re}\left\{-i\exp 2i\left[\int^{x_0}\frac{m\omega\,dx}{\sqrt{2m(E - I\omega)}} + \varphi_-\right]\right\}. \tag{4}$$

Here x_0 is the zero of the function $\omega(x)$.

Actually during the motion the adiabatic invariant is defined to within an amount α, and the formula (4) is true only apart from a factor of the order of unity multiplying the exponential function. To calculate this factor one would have to develop a perturbation theory. In view of the fact that the classical perturbation theory is extremely complicated, it is convenient to solve the quantum-mechanical problem first and then go over to the classical limit.

The Schrödinger equation for the Hamiltonian (1) is

$$\frac{1}{2m}(\Delta\psi - m^2\omega^2(x)\,y^2\psi) = -E\psi. \tag{5}$$

Here we have set $\hbar = 1$. To solve the problem, following reference 5, we introduce an orthogonal coordinate system in which the variables in Eq. (5)

*Everywhere in the text expressions of the type $\int^x f(u)du$ are to be understood as meaning $\int_{-\infty}^x [f(u) - f(-\infty)]du + xf(-\infty)$.

"almost separate"

$$\xi = x - y^2\omega'/4\omega, \quad \eta = \sqrt{m\omega}\, y. \tag{6}$$

In these coordinates Eq. (5) takes the following form:

$$(L_0 + 2mE - m\omega\eta^2)\psi = -L_1\psi, \tag{7}$$

where

$$L_0 = m\omega\frac{\partial^2}{\partial\eta^2} + \sqrt{\omega}\frac{\partial}{\partial\xi}\left(\frac{1}{\sqrt{\omega}}\frac{\partial}{\partial\xi}\right); \tag{8}$$

$$L_1 = \eta^2\left\{(f')^2 + \frac{1}{2}f''f\right\}L_0 + \frac{1}{f^2}\left[(f')^2 - \frac{1}{2}f''f\right]\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial}{\partial\eta}\right)$$
$$-\frac{\eta^2}{f}\frac{\partial}{\partial\xi}\left[f\left((f')^2 - \frac{1}{2}f''f\right)\frac{\partial}{\partial\xi}\right] + O(\alpha^4), \qquad f = (m\omega)^{-1/2}. \tag{9}$$

2. THE ZEROTH APPROXIMATION

The zeroth-approximation is

$$\left[m\omega\left(\frac{\partial^2}{\partial\eta^2} - \eta^2\right) + \sqrt{\omega}\frac{\partial}{\partial\xi}\left(\frac{1}{\sqrt{\omega}}\frac{\partial}{\partial\xi}\right) + 2mE\right]\psi = 0. \tag{10}$$

Separating the variables in this equation, we get the set of solutions:

$$|n, E\rangle = Y_n(\eta) Z_{nE}(\xi),$$

where $Y_n(\eta)$ are the normalized eigenfunctions of an oscillator with frequency 1. It must be noted that there are three different cases, in which different expressions hold for the function $Z_{nE}(\xi)$:

1) The motion of the particle is infinite in both directions. In this case $Z_{nE}(\xi)$ has the form

$$Z_{nE}(\xi) = \left(\frac{m}{2\pi k_{nE}}\right)^{1/2}(m\omega)^{1/4}\left\{\exp\left(i\int^{\xi} k_{nE}\,d\xi\right)\right.$$
$$\left. + R_{nE}\exp\left(-i\int^{\xi} k_{nE}\,d\xi\right)\right\}, \tag{11}$$
$$k_{nE}^2 = 2mE - m\omega(n + 1/2).$$

2) There is one point on the real axis where $k_{nE} = 0$. This case corresponds to reflection from a "magnetic plug."

3) There are two real roots of the equation $k_{nE} = 0$. This corresponds to a particle that is in a "magnetic trap."

Only the first two cases are considered in the present paper. In the second case the function $Z_{nE}(\xi)$ has different forms on different sides of the turning point, and near the turning point it is a solution of the Airy equation. In this case, however, as Landau and Lifshitz have shown (cf. reference 6), one can use contour integration to calculate the matrix elements, and this leads to the same results as in the first case, which we shall treat here. The coefficient $k_{nE}(\xi)$ is of order of magnitude α on the interval where the important variation of ω occurs, and becomes exponentially small for $x \to +\infty$. The transverse quantum number n plays the role of the adiabatic invariant in the classical limit. In zeroth approximation there is no change of n.

3. THE SCATTERING MATRIX

Let us find the transition amplitude by perturbation theory

As is shown in reference 7, in calculating the matrix element $\langle n'E' | L_1 | nE\rangle$ we can keep only the first term in the expression for Z_{nE}. In the calculation of the matrix element one encounters integrals of the following type:

$$Q_{nE}^{n'E'} = \int_{-\infty}^{+\infty} \frac{\exp\{i(\rho + i\sigma)\}}{(k_{nE}k_{n'E'})^{1/2}}\frac{\omega'^2}{\omega^3}\,d\xi, \tag{12}$$

$$\rho + i\sigma \equiv \int^{\xi}(k_{nE} - k_{n'E'})\,d\xi. \tag{13}$$

The path of integration can be shifted into the half-plane in which $\sigma > 0$. In doing so one must carry the path around the singularities of the integrand. Then the branching associated with the vanishing of the quantity k_{nE} does not contribute to the required matrix element. In fact, near this point Eq. (12) loses its meaning; it must be replaced by a solution of Airy's equation which has no singularity at the point in question. The singularities of the integrand are the zeroes and poles of the function ω. The main contribution to the integral (12) will come either from the saddle point or else from the zero or pole ξ_0 of the function $\omega(\xi)$ that corresponds to the smallest value of $\sigma(\xi_0)$. For definiteness let us examine the case in which the contribution is either from the saddle point or from a simple zero.

At the saddle point ξ_1 we have $\omega(\xi_1) = (E - E')/(n - n')$. The points ξ_0 and ξ_1 lie on the curve $L[\mathrm{Im}\,\omega(\xi) = 0]$, which intersects the real axis at some point ξ'. If the quantity $(E - E')/(n - n')$ is positive, then ξ_1 lies on the curve L between the points ξ' and ξ_0. If, on the other hand, $(E - E')/(n - n') < 0$, then ξ_1 lies on the curve L beyond ξ_0. Let ω_m be the maximum value of $\omega(\xi)$ on the segment $(\xi'\xi_0)$ of the curve L. It is obvious that the condition for the existence of a saddle point is $(E - E')/(n - n') \le \omega_m$. Thus in the case $0 < (E - E')/(n - n') \le \omega_m$ the calculation of the integral (12) can be carried out by the ordinary method of steepest descents, which gives

$$Q_{nE}^{n'E'} = \left[\frac{\pi i}{(n-n')\,2mk_{nE}}\right]^{1/2} \frac{(\omega')^{3/2}}{\omega^3} \exp\{i\rho(\xi_1) - |\sigma(\xi_1)|\}. \quad (14)$$

The expression (14) cannot be applied in the cases $E = E'$ and $n = n'$. Indeed, in these cases the main contribution to the integral (12) comes from the residue at the zero ξ_0 of the function $\omega(\xi)$, which is a third-order pole of the integrand. In the case $E = E'$ the pole coincides with the saddle point. In the case $n = n'$ the pole does not coincide with the saddle point, but gives a larger contribution to the expression (12). Calculating the residue at ξ_0, we get

$$Q_{nE}^{n'E} = \pi \frac{n-n'}{4E} \exp\{i\rho(\xi_0) - |\sigma(\xi_0)|\}, \quad (15)$$

$$Q_{nE}^{nE'} = -\frac{\pi i}{A} \frac{(2m)^{1/2}}{(EE')^{1/4}} (\sqrt{E} - \sqrt{E'})^2 \exp\{i\rho(\xi_0) - |\sigma(\xi_0)|\}, \quad (16)$$

$$A = \omega'(\xi_0).$$

We recall that $\rho(\xi_0)$ and $\sigma(\xi_0)$ are taken from the definition (13) with the suitable values of nE, $n'E'$.

For

$$(E - E')/E \sim A^{1/2}(mE^3)^{-1/4} \sim \sqrt{\alpha}$$

the expression (14) goes over into the form (15). In the case $(E - E')/(n - n') < 0$ the integral (12) is given by the residue at the point ξ_0, which gives a result that coincides with Eq. (16). Using Eqs. (14), (15), (16) and the well known expressions for the matrix elements of an oscillator, we get

$$\langle n+2, E | L_1 | n, E\rangle = \frac{3\pi}{16} \sqrt{(n+1)(n+2)} \quad \exp\{i\rho(\xi_0) - |\sigma(\xi_0)|\},$$

$$\langle n-2, E | L_1 | n, E\rangle = \frac{3\pi}{16} \sqrt{(n-1)n} \exp\{-i\rho(\xi_0^*) - |\sigma(\xi_0)|\}. \quad (17)$$

For transitions with changes of n by more than 2 the results are exponentially small in comparison with those just stated.

For $(E - E')/(n - n') \gg \alpha^{1/2}$ the matrix element has the structure (14) with a somewhat different coefficient of the exponential function, which coefficient, however, is less than or of the order of unity everywhere in the domain of applicability.

It is shown in the Appendix that the subsequent approximations make contributions to the scattering matrix that are small in α. Thus the non-diagonal transition matrix elements reduce to the expressions (17). In virtue of the unitarity relation the diagonal element of the scattering matrix is of the form $a_{nn} = e^{i\varphi}[1 + 0(\exp\{-2|\sigma(\xi_0)|\}]$, where φ is a small quantity, $\varphi \lesssim \alpha$.

Knowing the scattering matrix, one can easily calculate the change of the adiabatic invariant of the particle. For this it is necessary to construct a wave packet describing the classical particle on its trajectory. Carrying out calculations analogous to those done in reference 3, we get

$$\frac{\Delta I}{I} = \left(\frac{3\pi}{8}\right) 2\,\mathrm{Re}\left[-i \exp 2i\left\{\int^{\xi_0} \frac{m\omega\, d\xi}{\sqrt{2m(E - I\omega)}} + \varphi_-\right\}\right]. \quad (18)$$

We note that the coefficient of the exponential function does not depend on I and E. Within the framework of our method it is not hard to take into account the curvature of the lines of force. This problem, however, needs further investigation.

APPENDIX

Let us estimate the contribution to elements of the scattering matrix made by the second approximation. Having in mind the application to the classical case, we confine ourselves to just the calculation of the matrix element $a_{n,\, n+2}$ (cf. reference 6). The second approximation of perturbation theory is given by the formula

$$\int \frac{\langle n, E | L_1 | n+2, E'\rangle \langle n+2, E' | L_1 | n+2, E\rangle + \langle n, E | L_1 | n, E'\rangle \langle E', n | L_1 | n+2, E\rangle}{E' - E - i\delta} dE'. \quad (A.1.)$$

Let us consider for definiteness the first term. We divide the integration over E' into three intervals: $(0, E)$, $(E, E + 2\omega_m)$ and $(E + 2\omega_m, \infty)$. In the interval $(0, E)$ the integrand is of the form [cf. Eq. (16)]

$$\frac{(\sqrt{E} - \sqrt{E'})^4}{A^2\sqrt{EE'}\,(E'-E)} \exp\left\{i \int^{\xi_0} (k_{nE} + k_{n+2,\,E} - 2k_{n+2,\,E'})\, d\xi\right\}. \quad (A.2)$$

The integrand has a triple zero at the point $E' = E$, and therefore the problem of passage around a singularity does not arise in the integration. At this same point the exponent has a maximum value which is equal to the exponent obtained in the first approximation. Therefore the main contribution comes from a region of the order α near $E' = E$. Integration over this region gives a coefficient for the exponential of the order α. Actually the expression (A. 2) ceases to be valid in a neighborhood of $E' = E$ of the order $\alpha^{1/2}$. For the estimate in this region we replace the matrix element

$\langle n, E | L_1 | n+2, E \rangle$ by the quantity (17). In this region the matrix element $\langle n+2, E' | L_1 | n+2, E \rangle$ is at least of the order α. It is not hard to see that in this case the integration in Eq. (A. 1) gives for the exponential function a coefficient of the order of or smaller than $\alpha^{1/2}$.

In this same way it can be shown that the integration over the interval $(E, +\infty)$ leads to this same estimate.

It is not hard to carry out analogous arguments for the subsequent approximations of perturbation theory.

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