SPECTRA OF K_{e4} AND $K_{\mu4}$ DECAYS

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We have obtained the electron and π-meson energy spectra and the $\pi - e$ angular correlation in K-meson decays $K \to 2\pi + e + \nu$, and also the μ-meson spectrum and the spectrum of the effective mass of the two π mesons in the $K \to 2\pi + \mu + \nu$ decay. The $K_{\mu4}$ decay probability is one order of magnitude smaller than that of the K_{e4} decay.

IN a paper dealing with $K \to 2\pi + e + \nu$ decays, Okun' and the author[1] calculated the probability and obtained the effective-mass spectrum of the two π mesons. The present paper is devoted to a further study of K_{e4} decays, and also of $K_{\mu4}$ decays. In Secs. 1 — 3 we calculate the energy spectra of the electrons and π mesons, and the angular $\pi - e$ distribution in the K_{e4} decay. In Secs. 4 and 5 we give the effective-mass spectra of two π mesons and the energy spectrum of the μ mesons in $K_{\mu4}$ decay. The last section contains an expression for the K_{e4} and $K_{\mu4}$ decay probabilities.

1. ENERGY SPECTRUM OF THE ELECTRONS IN K_{e4} DECAY

The relativistically-invariant amplitude of K_{e4} decay can be written in the form

$$M = \frac{\sqrt{2}G}{\sqrt{8E_K E_1 E_2}} \Big\{ f_1 Q_\sigma + f_3 q_\sigma + f_2 R_\sigma + \frac{f_4}{\mathfrak{M}^2} \varepsilon_{\sigma\rho\tau\eta} q_\rho (k_1)_\tau (k_2)_\eta \Big\} \bar{u}_e \gamma_\sigma (1 + \gamma_5) u_\nu, \quad (1)$$

where $k_1 = \{\mathbf{k}_1, E_1\}$ and $k_2 = \{\mathbf{k}_2, E_2\}$ are the 4-vectors of the π-meson momenta, q is the 4-momentum of the K meson, $Q = k_1 + k_2$, and $R = k_1 - k_2$; $\mathfrak{M}$ is a certain quantity with dimension of mass. The coefficients f_i are functions of the scalar products $(k_1 q)$, $(k_2 q)$, and $(k_1 k_2)$. We shall assume them to be slowly-varying quantities and treat them as constants in the calculations. Then, if we use the rule $\Delta T = 1/2$ for lepton decays of strange particles, it is easy to verify that the first two terms in (1), which are symmetrical with respect to the substitution $k_1 \to k_2$, correspond to a decay in which the π mesons produced are in a state with total isotopic spin $T = 0$, while the remaining terms, which reverse their sign when the momenta k_1 and k_2 are interchanged, correspond to a state $T = 1$. Calculations show that the contribution of the last term of the total decay probability is related to the contribution of the second term, (which also corresponds to a π-meson system state with $T = 1$,) approximately as $(f_4^2/f_2^2) \times (m_\pi/\mathfrak{M})$.[4]

Inasmuch as the K-meson decay proceeds via $p - \bar{\Lambda}$ pair production,* we can regard the mass $\mathfrak{M}$ as equal to the baryon mass. Then the last term in (1) is small and will be neglected henceforth, and we shall consider only the vector part of the baryon current in the expression (1).

If we neglect the interaction of the π mesons in the final state, then f_1, f_2, and f_3 are real, and the differential probability has the form

$$dw = \frac{G^2}{2(2\pi)^8 E_K} [2(Vp_L)(Vp_\nu) - V^2(p_L p_\nu)] \times \delta^4(q - k_1 - k_2 - p_L - p_\nu) \frac{d^3k_1}{E_1}\frac{d^3k_2}{E_2}\frac{d^3p_L}{E_L}\frac{d^3p_\nu}{E_\nu}. \quad (2)$$

In the case of K_{e4} decay, the vector V contains only the π-meson momenta:

$$V = f(k_1 + k_2) + f_2(k_1 - k_2). \quad (3)$$

Integration of (2) is best carried out by introducing the variables $Q = k_1 + k_2$ and $R = k_1 - k_2$. Then we see from Eq. (A1) of the appendix that after integration over the π-meson momenta we obtain

$$dw = G^2 M (2\pi)^{-5} \{ f^2 (1 + \cos\theta) Y^{1/2} + \tfrac{1}{3} f_2^2 M^{-2} [4M(M - E_e + E_e\cos\theta) - M^2(1 + \cos\theta) - 4E_\nu(1 - \cos\theta) \times (M - E_e + E_e\cos\theta)] Y^{3/2} \} E_e^2 dE_e \, d\cos\theta \cdot E_\nu^2 dE_\nu,$$

$$Y = [(M^2 - 2ME_e - 4m^2)/2(M - E_e + E_e\cos\theta) - E_\nu] \times [(M^2 - 2ME_e)/2(M - E_e + E_e\cos\theta) - E_\nu]^{-1}, \quad (4)$$

where M is the K-meson mass, m the π-meson mass, and θ the angle between the emission directions of the electron and of the neutrino. Inte-

*We use the Sakata model.[3]

grating over E_ν between the limits

$$0 \leqslant E_\nu \leqslant (M^2 - 2ME_e - 4m^2)/2(M - E_e + E_e \cos\theta)$$

and then integrating with respect to $\cos\theta$ from -1 to 1, we obtain the electron spectrum, which is more conveniently expressed by introducing the variable $x = (M^2 - 2ME_e)/4m^2$, which depends linearly on E_e:

$$w(x)\,dx = \frac{G^2 m^{10}}{3\pi^5 M^3}\left(\frac{M^2}{4m^2} - x\right)^2 \times \Big\{ f^2\Big[\Big(x + \frac{5}{4} - \frac{3}{8x}\Big)\sqrt{x(x-1)} - 3\Big(x - \frac{1}{2} + \frac{1}{8x}\Big)\ln(\sqrt{x} + \sqrt{x-1})\Big] + \frac{f_2^2}{2}\Big[\Big(x + 6 - \frac{5}{8x} + \frac{3}{16x^2}\Big)\sqrt{x(x-1)} - 3\Big(2x + \frac{1}{4x} - \frac{1}{16x^2}\Big)\ln(\sqrt{x} + \sqrt{x-1})\Big]\Big\}\,dx; \qquad (5)$$

the variation of x is within the limits $1 \le x \le (M/2m)^2$.

The distribution (5) has the form

$$w(x) = \frac{G^2 m^{10}}{3\pi^5 M^3}\,\frac{3}{8}\,[f^2 F_0(x) + f_2^2 F_1(x)], \qquad (5')$$

where the first term corresponds to a decay in which the π mesons are in a state with isotopic spin $T = 0$, while the second term corresponds to the case $T = 1$. The electron spectra (see Fig. 1) have in both cases a maximum at an electron energy ~ 70 Mev. The electron spectrum does not contain an interference term proportional to ff_2, because the term that describes the interference of the states of the system of π mesons of different parity drops out in the integration over the momenta of the π mesons.

The energy spectrum of the electrons in K_{e4} decay was also calculated by Chadan and Oneda.[4] The curves given in their article coincide with those obtained in the present paper.

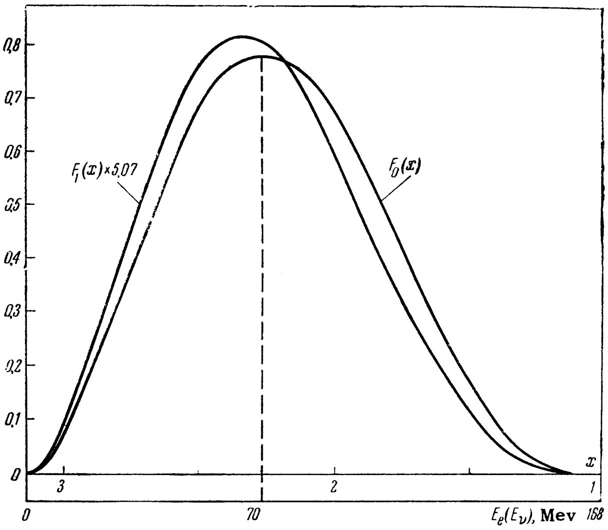

FIG. 1. Functions $F_0(x)$ and $F_1(x)$, which enter into formula (5) for the energy spectrum of the electrons in K_{e4} decay. The ordinate of the function F_1 is increased by a factor 5.07.

2. ENERGY SPECTRUM OF π MESONS IN K_{e4} DECAY

In order to obtain the spectrum of the π mesons and at the same time the $\pi - e$ angular correlation, we integrate (2) with respect to d^3k_1 and d^3p_ν. We do this once more by an invariant method, introducing the 4-vectors

$$F = k_2 + p_\nu, \quad L = k_2 - p_\nu. \qquad (6)$$

Neglecting the electron mass, we can write V in the form

$$V = fq + f_2(q - 2k_1). \qquad (7)$$

Substituting (7) in (2) and using Eq. (A2) of the appendix, we obtain

$$w(E_1, E_e, \cos\theta)\,dE_1\,d\cos\theta\,dE_e = 4\pi^3 G^2 M (2\pi)^{-8} k_1\,dE_1\,d\cos\theta \cdot \{f^2[M - E_1 - k_1\cos\theta - 2E_e] + f_2^2[M - E_1 - k_1\cos\theta - 2E_e - 4M^{-2}(E_1 - k_1\cos\theta) \times (M^2 - 2ME_1 + m^2 - 2E_e(M - E_1 + k_1\cos\theta))] + 2ff_2[M - E_1 + k_1\cos\theta + 2(m^2/M - E_1) - 2E_e(2M^{-1}(M - E_1 + k_1\cos\theta) - 1)]\} \times \Big\{\Big[\frac{M^2 - 2ME_1}{2(M - E_1 + k_1\cos\theta)} - E_e\Big] \Big/ \Big[\frac{M^2 - 2ME_1 + m^2}{2(M - E_1 + k_1\cos\theta)} - E_e\Big]\Big\}^2 E_e^2\,dE_e. \qquad (8)$$

We integrate the resultant expression further with respect to the electron energy, between the limits

$$0 \leqslant E_e \leqslant (M^2 - 2ME_1)/2(M - E_1 + k_1\cos\theta)$$

and then with respect to $\cos\theta$ from -1 to 1. The π-meson spectrum expressed in terms of the variable $y = (M^2 - 2ME_1 + m^2)/m^2$ has the form

$$w_\pi(y)\,dy = \frac{G^2\pi^3 m^7}{2(2\pi)^8}\,\frac{k}{m}\,\frac{1}{y^2}\Big\{f^2\Big[I_1 y - I_2 + 2\frac{k^2}{m^2}\Big(I_1 - \frac{2I_2}{3y}\Big)\Big] + f_2^2\Big[\Big[1 - \frac{2m^2}{M^2}\Big(\frac{M^2 - m^2}{m^2} - y\Big)\Big] \times (I_1 y - I_2) + 2\frac{k^2}{m^2}\Big(I_1 - \frac{2}{3}\frac{I_2}{y}\Big)\Big] + 2ff_2\Big[\Big[y + \frac{m^2}{2M^2}\Big(y^2 - \Big(\frac{M^2 - m^2}{m^2}\Big)^2\Big)\Big] I_1 - I_2\frac{m^2}{M^2 y}\Big[y - \frac{1}{4}\Big(y - \frac{M^2 - m^2}{m^2}\Big)^2 - \frac{M^2 k^2}{3m^4}\Big]\Big]\Big\}\,dy, \qquad (9)$$

where $k = |\mathbf{k}|$ is the π-meson momentum;

$$I_1 = (y-1)^3/3 + 2y\,[2(y-1)-(y+1)\ln y],$$

$$I_2 = (y-1)^4/4 + (28y^3-27y^2-1)/6 - y^2(3-2y)\ln y; \quad (10)$$

y varies in the limits $1 \leq y \leq (M-m)^2/m^2$.

We represent (9) in the form

$$w(y) = \frac{G^2\pi^3 m^7}{2(2\pi)^8}\,[f^2\Phi_0(y) + f_2^2\Phi_1(y) + 2ff_2\Phi(y)]. \quad (9')$$

The first term corresponds to K_{e4} decay in which the π mesons are in the state T = 0, the second corresponds to T = 1 and the third describes the interference of these states. The interference term is negative at large π-meson energies and positive at low energies, so that its integral vanishes. This is tantamount to stating that the total probability does not contain a term proportional to ff_2. The spectra of the π mesons (see Fig. 2) formed in the pure states T = 0 or T = 1 have broad maxima at approximately 25 Mev.

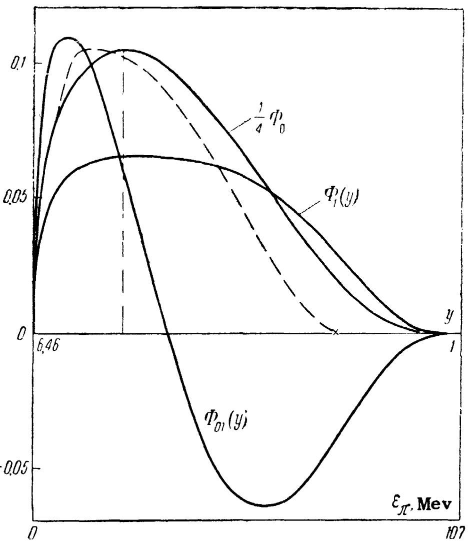

FIG. 2. The functions (¼) $\Phi_0(y)$, $\Phi_1(y)$, and $\Phi_{01}(y)$, which enter into the expression (9′) for the energy spectrum of the π mesons in K_{e4} decay; y varies in the limits $1 \leq y \leq 6.46$, corresponding to a π-meson kinetic-energy interval 107 Mev $\geq \varepsilon_\pi \geq 0$.

In the decay $K^+ \to \pi^+ + \pi^- + e^+ + \nu$, in which there is a mixture of the states T = 0 and T = 1, the presence of the interference term can cause a change in the form of the spectrum. In particular, if the sign of the interference term is negative, the maximum should shift towards larger π-meson energies. Therefore a study of the energy spectrum of the π mesons could possibly yield information on the relative signs of the constants f and f_2.

The spectrum of the π mesons in K_{e4} decay was also calculated by Mathur,[2] who used the nonrelativistic approximation. The π-meson spectrum obtained in the present work (and also the value of the maximum π-meson energy) differs from the result contained in Mathur's paper. For comparison, the spectrum obtained by Mathur for a decay in which the π mesons have T = 0 is shown dashed in Fig. 2.

3. ANGULAR π – E CORRELATION IN K_{e4} DECAY

The angular correlation between the directions of emission of the π meson and the electron is obtained from (8) by integration over E_e and then over E_1 — the π-meson energy. Integration over E_e yields

$$\begin{aligned} w(E_1, \cos\theta)\,dE_1\,d\cos\theta &= \frac{G^2\pi^3 k_1 M m^6\,dE_1}{2(2\pi)^8}\Big\{f^2\Big[\frac{M-E_1-k_1\cos\theta}{(M-E_1+k_1\cos\theta)^3}I_1 \\ &- \frac{I_2 m^2}{(M-E_1+k_1\cos\theta)^4}\Big] + f_2^2\Big[\frac{M-E_1-k_1\cos\theta}{(M-E_1+k_1\cos\theta)^3}I_1 \\ &- \frac{I_2 m^2}{(M-E_1+k_1\cos\theta)^4} - \frac{4}{M^2}\frac{(E_1-k_1\cos\theta)}{(M-E_1+k_1\cos\theta)^3} \\ &\times((M-2ME_1+m^2)I_1 - I_2 m^2)\Big] \\ &+ 2ff_2\Big[\frac{M-3E_1+k_1\cos\theta+2m^2M^{-1}}{(M-E_1+k_1\cos\theta)^3}I_1 \\ &- \frac{M-2E_1+2k_1\cos\theta}{M(M-E_1+k_1\cos\theta)^4}I_2 m^2\Big]\Big\}\,d\cos\theta. \end{aligned} \quad (11)$$

Here I_1 and I_2 are determined by (10). Integrating (numerically) with respect to the energy E_1, we obtain the distribution over the angle between the directions of π and e. The angular distribution has the form

$$w(\cos\theta) = \frac{G^2\pi^3 m^7}{4(2\pi)^8}\,[f^2\varphi_0(\cos\theta) + ff_2\varphi_{01}(\cos\theta) + f_2^2\varphi_1(\cos\theta)]; \quad (11')$$

and is shown in Fig. 3. It is seen from the figure

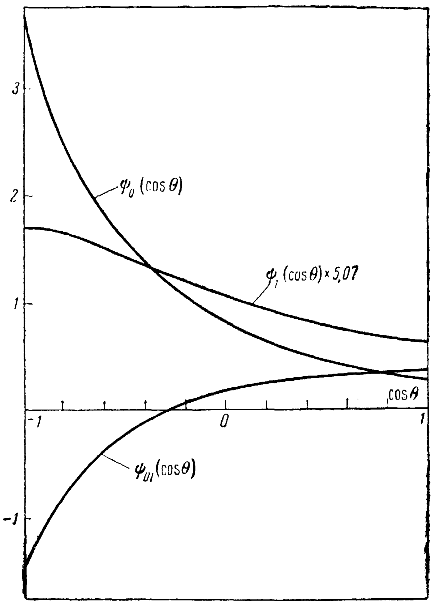

FIG. 3. Distribution over the angle between the directions of emission of the π meson and the electron (or neutrino) [Eq. (11′)].

that in the cases T = 0 and T = 1 the electron and the π meson are scattered predominantly at angles greater than $\pi/2$. Interference may change the form of the spectrum.

4. ANGULAR SPECTRUM OF μ MESONS IN $K_{\mu 4}$ DECAY

In the case of $K_{\mu 4}$ decay, the vector V in Eq. (2) contains three phenomenological constants:

$$V = f_1(k_1 + k_2) + f_2(k_1 - k_2) + f_3 q. \quad (12)$$

As in the calculation of the electron spectrum in the K_{e4} decay, we introduce $Q = k_1 + k_2$ and $R = k_1 - k_2$ and integrate over $d^4Q\, d^4R$. We obtain

$$dw = \frac{G^2}{M(2\pi)^5}\Big\{f_1^2[M^2(E_\mu + p_\mu \cos\theta) - 2M\mu^2 + \mu^2(E_\mu - p_\mu\cos\theta)] + f_3^2 M^2(E_\mu + p_\mu\cos\theta) + 2f_1f_3[M^2(E_\mu + p_\mu\cos\theta) - \mu^2 M] + \frac{1}{3}f_2^2[2M^2E_\mu - 2\mu^2 M + (E_\mu - p_\mu\cos\theta)(M^2 - 4ME_\mu + 3\mu^2) - 4E_\nu(E_\mu - p_\mu\cos\theta)(M - E_\mu + p_\mu\cos\theta)]\frac{Q^2 - 4m^2}{Q^2}\Big\} \times \left(\frac{Q^2 - 4m^2}{Q^2}\right)^{1/2} p_\mu\, dE_\mu\, d\cos\theta\, E_\nu^2\, dE_\nu, \quad (13)$$

$$Q^2 = M^2 - 2ME_\mu + \mu^2 - 2(M - E_\mu + p_\mu\cos\theta)E_\nu. \quad (14)$$

Integrating with respect to E_ν in the limits

$$0 \leqslant E_\nu \leqslant \frac{M^2 - 2ME_\mu + \mu^2}{2(M - E_\mu + p_\mu\cos\theta)}$$

and with respect to $\cos\theta$ in the limits $-1 \le \cos\theta \le 1$, we obtain the spectrum of the μ mesons in $K_{\mu 4}$ decay. Expressed in terms of the variable $\kappa = (M^2 + \mu^2 - 2ME_\mu)/4m^2$, this spectrum has the form

$$w(\kappa)\,d\kappa = \frac{G^2 m^{10}}{3\pi^5 M^3}\left\{\left[\left(\frac{M-\mu}{2m}\right)^2 - \kappa\right]\left[\left(\frac{M+\mu}{2m}\right)^2 - \kappa\right]\right\}^{1/2} \times \Big\{f_1^2\kappa\left(\frac{M^2-\mu^2}{4m^2} - \kappa\right)J_1(\kappa) + 2f_1f_3\kappa\left(\frac{M^2+\mu^2}{4m^2} - \kappa\right)J_1(\kappa) + f_3^2\left[\frac{M^2}{4m^2}\left(\frac{M^2-\mu^2}{4m^2} - \kappa\right) - \left(\frac{(M-\mu)^2}{4m^2} - \kappa\right) \times \left(\frac{(M+\mu)^2}{4m^2} - \kappa\right)\right]J_1(\kappa) + \frac{f_2^2}{2}\left(\frac{M^2-\mu^2}{4m^2} - \kappa\right)J_2(\kappa)\Big\}\,d\kappa;$$

$$J_1(\kappa) = \left(1 + \frac{5}{4\kappa} - \frac{3}{8\kappa^2}\right)\sqrt{\kappa(\kappa-1)} - 3\left(1 - \frac{1}{2\kappa} + \frac{1}{8\kappa^2}\right)\ln(\sqrt{\kappa} + \sqrt{\kappa-1}),$$

$$J_2(\kappa) = \left(\kappa + 6 - \frac{5}{8\kappa} + \frac{3}{16\kappa^2}\right)\sqrt{\kappa(\kappa-1)} - 3\left(2\kappa + \frac{1}{4\kappa} - \frac{1}{16\kappa^2}\right)\ln(\sqrt{\kappa} + \sqrt{\kappa-1}). \quad (15)$$

The variable κ ranges from 1 to $(M - \mu)^2/4m^2$. As $\mu \to 0$, the meson spectrum goes into the electron spectrum, with

$$(f_1 + f_3)^2 \to f^2, \quad f_2^2 \to f_2^2.$$

We can represent (15) in the form

$$w(\kappa)\,d\kappa = \frac{G^2 m^{10}}{30\pi^5 M^3}\{f_1^2 F_0^{(1)}(\kappa) + 2f_1f_3F_0(\kappa) + f_3^2F_0^{(3)}(\kappa) + f_2^2F_1^{(2)}(\kappa)\}\,d\kappa. \quad (15')$$

Plots of the functions $F_\alpha^{(i)}$ are shown in Fig. 4.

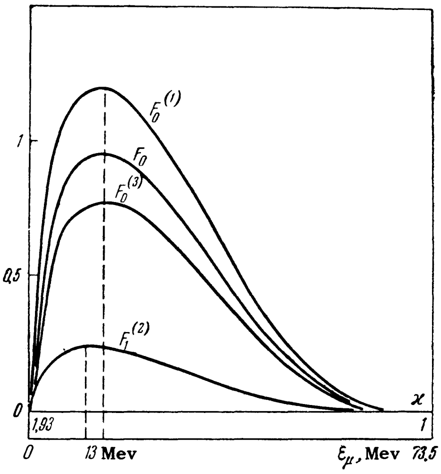

FIG. 4. The functions $F_\alpha^{(i)}(\kappa)$, which enter into the expression (15′) for the energy spectrum of μ mesons in $K_{\mu 4}$ decay (ε_μ is the kinetic energy of the μ meson).

The spectra have a maximum at a μ-meson energy 13 Mev.

5. EFFECTIVE MASS SPECTRUM OF TWO π MESONS IN $K_{\mu 4}$ DECAY

The effective mass of the two π mesons is

$$Q_{eff} = \sqrt{Q^2} = \sqrt{(E_1 + E_2)^2 - (\mathbf{k}_1 + \mathbf{k}_2)^2}.$$

Substituting (12) in (2) and using the appendix formulas (A1) and (A3) we obtain

$$dw = \frac{G^2}{M(2\pi)^8}\sqrt{\frac{Q^2 - 4m^2}{Q^2}\frac{(F^2-\mu^2)^2}{F^4}}\Big\{\frac{(f_1+f_3)^2}{3F^2}[(F^2+2\mu^2) \times (QF)^2 - (2F^2+\mu^2)Q^2F^2] + 2(f_1+f_3)f_3\mu^2(QF) + f_3^2\mu^2F^2 + \frac{f_2^2}{9}\frac{Q^2-4m^2}{Q^2}\Big[2\frac{F^2+2\mu^2}{F^2}(QF)^2 + (4F^2 - \mu^2)Q^2\Big]\Big\}\sqrt{Q_0^2 - Q^2}\,Q\,dQ\,dQ_0\,d\Omega; \quad (16)$$

here $F = q - Q$ and Q are 4-vectors. Integration over $d\Omega$ yields 4π, while integration over dQ_0 in the limits $Q \le Q_0 \le (Q^2 + M^2 - \mu^2)/2M$ leads to the following result: if we put $Q^2/M^2 = z$, then the spectrum over z is given by

$$w(z)\,dz = \frac{G^2M^7\pi^3}{6(2\pi)^8}\sqrt{1-4m^2/M^2z}\,\{(f_1+f_3)^2A(z)+f_3^2B(z)$$
$$+f_3(f_1+f_3)D(z)+\tfrac{1}{3}f_2^2[A(z)$$
$$+6zB(z)M^2/\mu^2]\,[1-4m^2/M^2z]\}\,dz. \qquad (17)$$

Here

$$A(z)=\tfrac{1}{8}[1-11\alpha-47\alpha^2-3\alpha^3+(-7+32\alpha+13\alpha^2)z$$
$$+(-7+\alpha)z^2+z^3]\sqrt{(1-\alpha+z)^2-4z}$$
$$+3\left[-\tfrac{3}{2}\alpha^2-\alpha^3+2\alpha^2z+(1-2\alpha+\alpha^2/2)z^2\right]\ln(\mathrm{I})$$
$$+3\alpha^2\left[-\tfrac{3}{2}-\alpha+(2z-z^2/2)\right]\ln(\mathrm{II}),$$

$$B(z)=\tfrac{1}{2}\alpha[1-5\alpha-2\alpha^2+5(2-\alpha)z$$
$$+z^2]\sqrt{(1-\alpha+z)^2-4z}-6\alpha[\alpha^2/2$$
$$+(1-2\alpha+\alpha^2/2)z+z^2]\ln(\mathrm{I})-3(1-z)\alpha^3\ln(\mathrm{II}),$$

$$D(z)=\alpha[1-5z-2z^2+5\alpha(2-z)$$
$$+\alpha^2]\sqrt{(1-\alpha+z)^2-4z}$$
$$+6\alpha[\alpha+\alpha^2-2\alpha z+(2-\alpha)z^2]\ln(\mathrm{I})$$
$$+6\alpha^2[\alpha+(1-z)^2]\ln(\mathrm{II}),$$

where $\alpha = \mu^2/M^2$ and we introduce the notation

$$\ln(\mathrm{I})=\ln\frac{2\sqrt{z}}{1-\alpha+z-[(1-\alpha+z)^2-4z]^{1/2}},$$

$$\ln(\mathrm{II})=\ln\frac{2\sqrt{z}\alpha}{(1-z)[(1-\alpha+z)^2-4z]^{1/2}-(1-z)^2-(1+z)\alpha}.$$

The effective-mass spectrum can be represented in the form

$$w(z)=\frac{10^{-3}G^2M^7\pi^3}{6(2\pi)^8}\Big\{(f_1+f_3)^2A'(z)+3\frac{\mu^2}{M^2}f_3^2B'(z)$$
$$+3\frac{\mu^2}{M^2}f_3(f_1+f_3)D'(z)+\tfrac{1}{3}f_2^2H(z)\Big\}. \qquad (17')$$

This spectrum is shown in Fig. 5. We see that the most probable decays are those in which the π mesons carry away a small effective mass, amounting to 0.6 M_K and nearly equal to twice the π-meson mass (0.565 M_K).

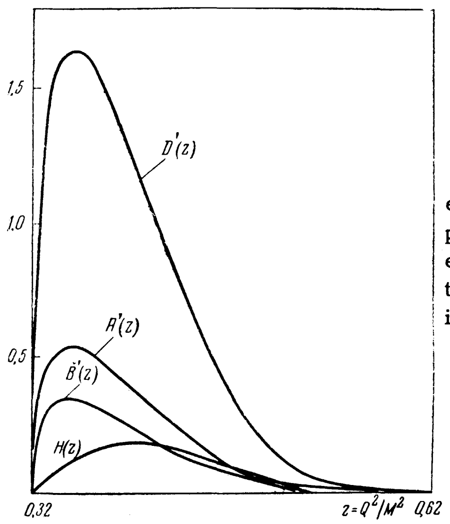

FIG. 5. Functions entering into the expression (17′) for the effective-mass spectrum of two π mesons in $K_{\mu 4}$ decay.

We note that the effective-mass spectrum of the π mesons produced in a state with isotopic spin T = 1 is shifted towards larger values of the effective mass. Accordingly, the electron spectra in K_{e4} decays and the μ-meson spectra in $K_{\mu 4}$ decays, pertaining to this case, have maxima at somewhat lower values of lepton energy than if the π mesons form a system with T = 0.

6. PROBABILITIES OF K_{e4} AND $K_{\mu 4}$ DECAYS

The matrix elements of the KL_2, KL_3, and KL_4 decays have the following form:

$$K^+_{e(\mu)2}:\quad (G/\sqrt{2})\,\varphi_K\varkappa j^L_\alpha q_\alpha,$$
$$K^+_{e(\mu)3}:\quad (G/\sqrt{2})\,\varphi_K\varphi_\pi j^L_\alpha(g_1q_\alpha+g_2k_\alpha),$$
$$K^+_{e(\mu)4}:\quad (G/\sqrt{2})\,\varphi_K\varphi_\pi\varphi_\pi j^L_\alpha(f_1'k_1+f_2'k_2+f_3q)_\alpha, \qquad (18)$$

where $j^L_\alpha = \bar{u}_\nu\gamma_\alpha(1+\gamma_5)u_L$. For decays such as $K^+ \to N\pi^0 + e^+ + \nu$, where N = 0, 1, 2, these matrix elements become

$$N=0:\quad (G/\sqrt{2})\,\varphi_K j^L_\alpha\varkappa q_\alpha,$$
$$N=1:\quad (G/\sqrt{2})\,\varphi_K\varphi_\pi j^L_\alpha(g_1+g_2)q_\alpha,$$
$$N=2:\quad (G/\sqrt{2})\,\varphi_K\varphi_\pi\varphi_\pi j^L_\alpha[(f_1'+f_2')/2+f_3]q_\alpha. \qquad (19)$$

From the known probabilities of $K_{\mu 2}$ and K_{e3} decays we can obtain

$$(g_1+g_2)^2/\varkappa^2 \approx 10^{-1}/0.14\cdot 10^{-2}M_p^2 \approx 72/M_p^2 \qquad (20)$$

(M_p is the proton mass).

Assuming that each "excess" π meson introduces the same dimensional factor (20), i.e., assuming (for the K_{e4} and K_{e3} decays)

$$[(f_1'+f_2')/2+f_3]^2/(g_1+g_2)^2=(g_1+g_2)^2/\varkappa^2=72/M_p^2,$$

we obtain for the branching ratio of the K_{e4} and K_{e3} decays a value

$$w(K^+\to 2\pi^0+e^++\nu)/w(K^+\to\pi^0+e^++\nu)\approx 0.8\cdot 10^{-3}.$$

The probability of $K_{\mu 4}$ decay is

$$w_{K_{\mu 4}}=\frac{G^2m^{10}}{3\pi^5M^3}[0.0334\,f_1^2+2\cdot 0.041\,f_1f_3$$
$$+0.051\,f_3^2+0.0046\,f_2^2].$$

If we let the μ-meson mass approach zero, we obtain for the probability of K_{e4} decay

$$w_{K_{e4}}=(G^2m^{10}/3\pi^5M^3)\,[0.3\,(f_1+f_3)^2+0.06\,f_2^2].$$

It is thus found that the probability of $K_{\mu 4}$ is one order of magnitude smaller than the probability of K_{e4} decay.

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APPENDIX

In the integration over k_1 and k_2 (the momenta of the π mesons) we introduce the variables $Q = k_1 + k_2$ and $R = k_1 - k_2$. Then

$$\frac{d^3k_1}{E_1}\frac{d^3k_2}{E_2} = \delta(QR)\,\delta(Q^2 + R^2 - 4m^2)\,d^4Q d^4R,$$

$$\int \delta(QR)\,\delta(Q^2 + R^2 - 4m^2)\,d^4R = 2\pi\,[(Q^2 - 4m^2)/Q^2]^{1/2},$$

$$\int (RB)^2\delta(QR)\,\delta(Q^2 + R^2 - 4m^2)\,d^4R$$

$$= \frac{2\pi}{3}\,[(QB)^2 - Q^2B^2]\left(\frac{Q^2 - 4m^2}{Q^2}\right)^{1/2}, \qquad \text{(A1)}$$

where B is any 4-vector.

Introducing the vectors $F = k_2 + p_\nu$ and $L = k_2 - p_\nu$, we obtain

$$\frac{d^3p_\nu}{E_\nu}\frac{d^3k_2}{E_2} = \delta(F^2 + L^2 - 2m^2)\,\delta(FL - m^2)\,d^4F d^4L,$$

$$\int \delta(F^2 + L^2 - 2m^2)\,\delta(FL - m^2)\,d^4L = 2\pi\,(F^2 - m^2)/F^2,$$

$$\int (Bp_\nu)\,\delta(F^2 + L^2 - 2m^2)\,\delta(FL - m^2)\,d^4L = \pi\,(BF)\left(\frac{F^2 - m^2}{F^2}\right)^2. \qquad \text{(A2)}$$

We denote by F and L the 4-vectors $F = p_\mu + p_\nu$ and $L = p_\mu - p_\nu$, where p_μ and p_ν are the 4-momenta of the μ meson and the neutrino. Then

$$\frac{d^3p_\mu}{E_\mu}\frac{d^3p_\nu}{E_\nu} = \delta(FL - \mu^2)\,\delta(F^2 + L^2 - 2\mu^2)\,d^4L d^4F,$$

$$\int \delta(F^2 + L^2 - 2\mu^2)\,\delta(FL - \mu^2)\,d^4L = 2\pi\,(F^2 - \mu^2)/F^2,$$

$$\int (NL)(ML)\,\delta(F^2 + L^2 - 2\mu^2)\,\delta(FL - \mu^2)\,d^4L$$

$$= 2\pi\frac{F^2 - \mu^2}{F^2}\left\{\mu^4\frac{(FN)(FM)}{F^4} + \frac{(F^2 - \mu^2)^2}{3F^4}\,[(FN)^2 - F^2N^2]^{1/2}\,[(FM)^2 - F^2\mu^2]^{1/2}\right\}. \qquad \text{(A3)}$$

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