LOCATION OF THE NEAREST SINGULARITIES OF THE $\pi\pi$-SCATTERING AMPLITUDE

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A number of $\pi\pi$-scattering diagrams are considered for which the so-called singular curves (threshold of appearance of imaginary part of the amplitude) have asymptotic values equal to 16 μ^2.

A knowledge of the singular points of amplitudes describing physical processes is of interest for its own sake as well as in connection with attempts to describe strong interactions phenomenologically.[1]

We shall find the singular points of the $\pi\pi$-scattering amplitude by making use of perturbation theory Feynman diagrams.[2] It is obvious that in order to find nearest lying singular points it is sufficient to consider only pions as the virtual particles.

The singular points corresponding to diagrams 1 and 2 (see Fig. 1)* were determined by Mandelstam.[3] Denoting by W the total energy of the incident pions in their barycentric frame, and by Q the momentum transfer we have:

$$Q^2 = 4W^2/(W^2 - 16) \text{ (diagram 1)} \quad (1)$$

$$Q^2 = 16W^2/(W^2 - 4) \text{ (diagram 2)} \quad (2)$$

(W and Q are expressed in units of pion mass).

In the $W^2 - Q^2$ plane the curves 1 and 2 have as asymptotes the straight lines $Q^2 = 4$, $W^2 = 16$ and $Q^2 = 16$, $W^2 = 4$. It is obvious from the pseudoscalar nature of the pion that the next boundary of singularities curves will have in the $W^2 - Q^2$ plane the asymptotes $Q^2 = 4$ and $W^2 = 36$ ($Q^2 = 36$, $W^2 = 4$) or $Q^2 = 16$ and $W^2 = 16$. We restrict ourselves to discussion of diagrams belonging to the latter class.

We were able to find only the four diagrams (3 — 6, Fig. 1), for which the singular curves have as asymptotes 16 and 16. Diagram 3 was studied by us previously.[4] It is convenient to express the curve corresponding to this diagram in parametric form

$$W^2(x, y) = 2(1 - xy)[1 - 2/(x - y)]^2,$$
$$Q^2(x, y) = 2(1 + xy)[1 + 2/(x + y)]^2, \quad (3)$$

with the two parameters x and y related by

$$x^2 = -1/y + 1 + y. \quad (3')$$

Furthermore only the following values of x and y are permissible: $y > x$ and $y > -x$, as a consequence of the condition that the Feynman parameters α be positive. For $W^2 = Q^2$ we have $W^2 = Q^2 = 2(2 + \sqrt{5})^2 = 35.8$ and $x = 0$ while $y = (\sqrt{5} - 1)/2$.

The singular points for the diagrams 4 — 6 will be found by making use of the method described in detail previously.[4] The elementary, but occasionally tedious, steps will not be given. We make only two remarks. First, when the relation between Q^2 and W^2 is in parametric form it is convenient to use as parameters ratios of the Feynman parameters α, since the latter are positive and this facilitates the determination of the region of permissible values of the parameters. Second, by taking into account the symmetries in the Feynman diagrams, the determination of the location of the singularities is drastically simplified. This symmetry leads to the equality of the corresponding scalar products,* or, what is equivalent, to equalities among the α. We indicate below what equalities among the α resulting from the symmetry of Feynman diagrams were used by us.

For diagram 4 the symmetry leads to $\alpha_1 = \alpha_3$; $\alpha_2 = \alpha_4$; $\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8$. The singular curve is given parametrically as follows:

$$W^2(x) = (3 + x)(9 - x^2)/(1 + x),$$
$$Q^2(x) = (3 - x)(9 - x^2)/(1 - x). \quad (4)$$

The parameter x is restricted to the range $-1 \leq x \leq 1$ by the condition that α be positive. Here $W^2(-1) = \infty$, $Q^2(-1) = 16$ and $W^2(1) = 16$, $Q^2(1) = \infty$. For $x = 0$ $W^2(0) = Q^2(0) = 27$.

For the diagram 5 the symmetry results in the equality $\alpha_2 = \alpha_4$. The singular curve is given by

*In the diagrams referring to $\pi\pi$ scattering each pion is represented by a single line, while double and triple lines denote the exchange of two and three pions respectively.

*As shown by A. Z. Patashinskiĭ (private communication), one can verify directly that there are no nonsymmetric solutions for a majority of the diagrams considered here.

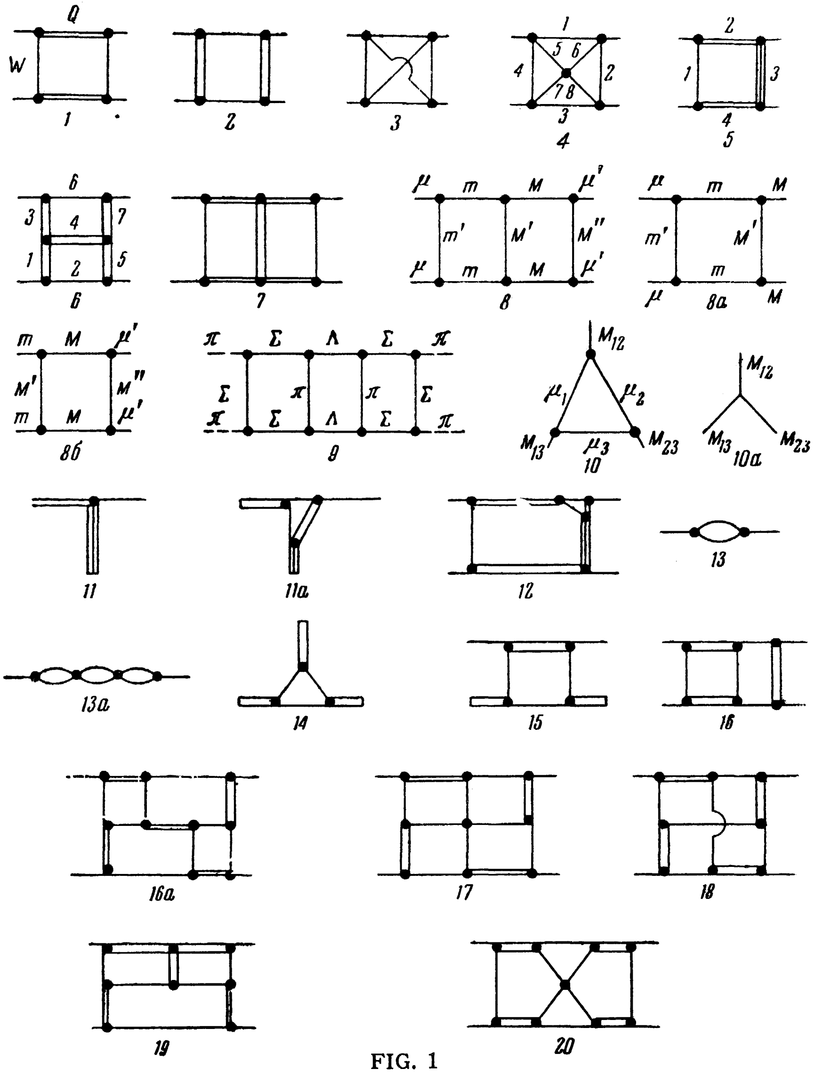

FIG. 1

$$(W^2 - 16)\,(Q^2 - 16) = 192. \tag{5}$$

It is obvious that the asymptotes are $W^2 = 16$ and $Q^2 = 16$. For $W^2 = Q^2$ we have $W^2 = Q^2 = 8(2 + \sqrt{3}) = 29.9$.

For the diagram 6 the symmetry leads to the equalities: $\alpha_2 = \alpha_6$; $\alpha_1 = \alpha_3 = \alpha_5 = \alpha_7$. The singular curve is given by

$$Q^2 = 16[\sqrt{W^2} - 1]^2 / \sqrt{W^2}\,[\sqrt{W^2} - 4]. \tag{6}$$

For $W^2 \to \infty$, $Q^2 \to 16$ and for $Q^2 \to \infty$, $W^2 \to 16$. For $W^2 = Q^2$ we have $W^2 = Q^2 = 34.6$.

The singular curves corresponding to the diagrams 1 — 6 are shown in Fig. 2 (curve 7 in Fig. 2 corresponds to diagram 7, which is obtained from 6 by the substitution $W^2 \rightleftarrows Q^2$).

As was already mentioned, we were unable to find any diagrams, other than diagrams 3 — 6, that would give rise to singular curves with asymptotes $Q^2 = 16$ and $W^2 = 16$. However we do not have a rigorous proof that no other such diagrams exist. We list below various considerations used by us in determining the absence of singular points in a number of diagrams.

First, we have assumed that it is sufficient to consider diagrams which when cut by an arbitrary horizontal or vertical line give a sum of the masses of the virtual particles equal to four. This assumption was based on the example of diagram 8 (the mass of the particle is given next to each line). In order that this diagram have a singular point it is necessary that diagrams 8a and 8b have singular points. If $\mu^2 < m^2 + m'^2$ and $M^2 < m^2 + M'^2$ then diagram 8a has a singularity only for $W^2 > 4m^2$. Analogously, if $m^2 < M^2 + M'^2$ and $\mu'^2 < M^2 + M''^2$ then diagram 8b has a singularity only for $W^2 > 4M^2$. Consequently diagram 8 has a singular curve with asymptotes $W^2 = 4\epsilon^2$ with ϵ the larger of the masses m and M.*

*Proceeding in an analogous manner, it is easy to show that if the above conditions on the squares of the masses are violated either for diagram 8a or diagram 8b, then diagram 8 has no singularities at all; diagram 9 is an example of such a case.

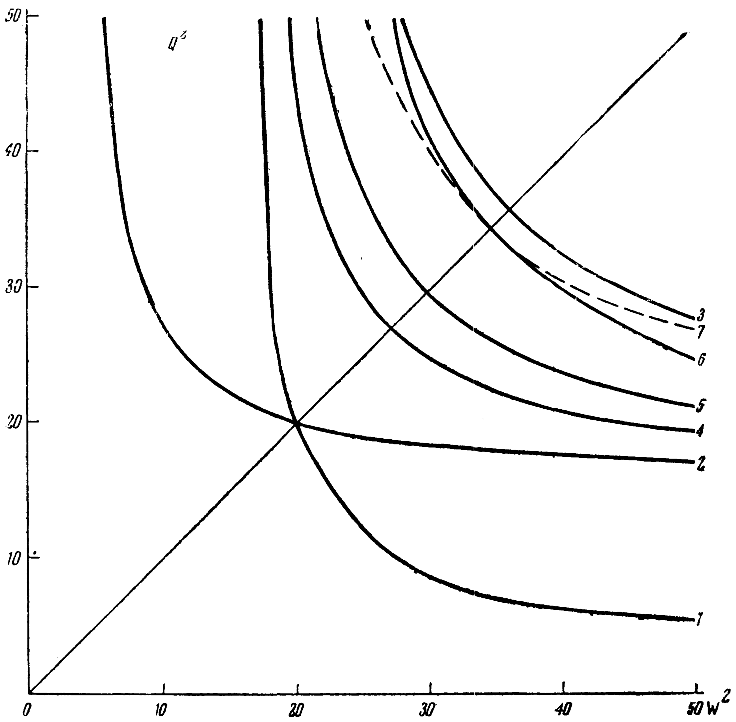

FIG. 2

Second, a substantial simplification in the analysis of complicated diagrams results when there is contained within them a three-sided contour such that the mass of the particle represented by the line entering one of the vertices is equal to the sum or the difference of the masses of the particles corresponding to the internal lines entering the same vertex. Diagram 10 is an example of such a contour. It is easy to verify that if $M_{12} = |\mu_1 - \mu_2|$, then it is necessary for the existence of a singular point to have $M_{13} = \mu_1 + \mu_3$ and $M_{23} = \mu_2 + \mu_3$. On the other hand, if $M_{12} = \mu_1 + \mu_2$, then $M_{13} = |\mu_1 \, n \, \mu_3|$ and $M_{23} = |\mu_2 \mp \mu_3|$. A three-sided contour satisfying these conditions can be transformed into a vertex (diagram 10a) and conversely the vertex of diagram 10a may be transformed into the three-sided contour of diagram 10.*

Let us discuss, for example, the vertex of diagram 11. It may be transformed into the three-sided contour, diagram 11a. In this manner it is possible to discuss along with diagram 5 diagram 12 which differs from diagram 5 by having one of the vertices satisfying the above conditions replaced by a three-sided contour. The location of the singularities of diagram 12 is the same as the location of the singularities of diagram 5. Applying to diagram 12 successive transformations of a vertex into a three-sided contour we obtain more and more new diagrams, all of which have singular lines coinciding with those of diagram 5. A situation arises that is analogous to what happens to the singularities of Green's functions: diagrams 13 and 13a have singularities in the same locations.

Let us also note the trivial circumstance that if one of the vertices of a three-sided contour satisfies the relation $M_{12} = |\mu_1 \pm \mu_2|$ while the other vertices fail to satisfy the corresponding relations, as for example is the case for diagram 14, then the presence of such a contour in a given diagram results in the absence of singularities.

Third, it is easy to establish that the four-sided contour of diagram 15 has no singular curves. As a result diagrams containing this four-sided contour also have no singularities (e.g., diagrams 16 and 16a).

Fourth, for a number of diagrams the absence of singularities was established by direct calculations (the impossibility of satisfying the condition that the parameters α be positive). Some of these diagrams (17—20) are shown in Fig. 1.

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[3] S. Mandelstam, preprint

[4] Kolkunov, Okun', and Rudik, JETP **38**, 877 (1960), Soviet Phys. JETP **11**, 634 (1960).

*It is essential here that only one relation among the α exists in a three-sided contour.

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