THE VAVILOV-CERENKOV EFFECT IN UNIAXIAL CRYSTALS

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The Vavilov-Cerenkov effect is considered for the case of an electric charge moving uniformly in an arbitrary direction relative to the optic axis of a uniaxial crystal. The shapes of the cones of normals for the ordinary and extraordinary waves are studied, and simple expressions are derived for the energy of the emitted waves.

INTRODUCTION

A general theory of the Cerenkov radiation in optically anisotropic media has been developed by Ginzburg.[1] Starting from the Maxwell equations for nonmagnetic transparent crystals, he obtained expressions for the angular distribution of the emitted energy. More detailed studies of the properties of the radiation have been made only for charged particles moving parallel or perpendicular to the optic axis in uniaxial crystals.† Quite recently Pafomov[4] has treated the more realistic case of a bounded uniaxial crystal, for which the transition radiation is superposed on the Cerenkov radiation.

Using the interference properties of the Cerenkov radiation, Frank[5,6] has investigated the conditions necessary for the occurrence of the radiation. In his Nobel lecture[5] it was shown for the first time that the direction of the light ray and the ray velocity play an important part in this connection.

In the present paper it is assumed that the crystal is transparent and has no gyrotropic properties. Its magnetic properties are given by a scalar magnetic permeability μ, and its electrical properties by the dielectric constant ϵ_{ik}, which is a symmetric tensor. In the coordinate system of the principal axes the components are positive, $\epsilon_{ii} \geq 0$.

Let a point charge move in the direction of the unit vector $\mathbf{r}$ with a speed $w = c\beta$. For the energy that it loses in the time $1/w$ we have the following formula:

$$S = \int d\omega \sum_{\lambda=1,2} S_{(\lambda)}(\omega), \qquad (1)$$

where the energy per unit frequency range emitted in waves of polarization λ is

$$S_{(\lambda)}(\omega) = \frac{e^2 c^2 \omega}{2\pi\mu} \int d\mathbf{n} \frac{(\mathbf{e}^{(\lambda)}\mathbf{r})^2}{v^4_{(\lambda)}} \delta\left(\mathbf{rn} - \frac{v_{(\lambda)}}{c\beta}\right). \qquad (2)$$

Here $d\mathbf{n}$ is an element of solid angle, $\mathbf{n}$ is the unit vector in the direction of propagation of the phase of the wave, $v_{(\lambda)} > 0$ is the speed of this propagation, and the $\mathbf{e}^{(\lambda)}$ are vectors in the direction of the electric field.

Let us introduce unit vectors $\mathbf{d}^{(\lambda)}$ directed along the electric displacement and having the following properties:

$$\mathbf{d}^{(\lambda)}\mathbf{d}^{(\nu)} = \delta_{\lambda\nu}, \quad \mathbf{nd}^{(\lambda)} = 0, \quad d_i^{(\lambda)} = \varepsilon_{ik} e_k^{(\lambda)}. \qquad (3)$$

These equations, together with the condition

$$\mathbf{e}^{(1)}\mathbf{d}^{(2)} = 0 \qquad (4)$$

determine the polarization vectors. The speeds of the corresponding waves are obtained from the formula

$$v^2_{(\lambda)} = c^2 \mu^{-1} \mathbf{e}^{(\lambda)}\mathbf{d}^{(\lambda)}. \qquad (5)$$

Since we shall be considering uniaxial crystals, we let the optic axis be along the axis x_1. For the components of dielectric-constant tensor, in diagonal form, we use the notations $\epsilon_{11} = \tau$, $\epsilon_{22} = \epsilon_{33} = \epsilon$. Let the charge move in the plane (x_1, x_2), which we shall call the principal plane. Then

$$\mathbf{r} \equiv (r_1, r_2, 0), \quad r_1^2 + r_2^2 = 1, \quad r_1 \geqslant 0, \quad r_2 \geqslant 0. \qquad (6)$$

We shall now investigate separately the properties of the ordinary and extraordinary waves. Instead of the indices $\lambda = 1, 2$ we introduce the notation $\lambda = o, e$.

THE ORDINARY WAVES

From Eqs. (3) — (5) we get for the polarization vector and speed of propagation of the ordinary

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†See the review article by Bolotovskiĭ[2] and the book by Jelley.[3]

waves

$$\mathbf{e}^{(o)} \equiv (0, n_3, -n_2)/\varepsilon\sqrt{1-n_1^2}, \qquad v_{(o)} = c/\sqrt{\varepsilon\mu}. \qquad (7)$$

By using these formulas we can rewrite (2) as

$$S_{(o)}(\omega) = \frac{e^2\omega\mu}{2\pi c^2}\int dn \frac{r_2^2 n_3^2}{1-n_1^2}\,\delta\left(\mathbf{n}\mathbf{r} - \frac{1}{\sqrt{\varepsilon\mu}\,\beta}\right). \qquad (8)$$

Here, just as in an isotropic medium, the energy will be emitted in directions that lie on a conical surface of circular cross section, with its axis parallel to the motion of the charge. The angle of emission $\Theta_{(o)}$ is given by the formula

$$\cos\Theta_{(o)} = \beta_o/\beta, \qquad (9)$$

in which the quantity

$$\beta_o = (\varepsilon\mu)^{-1/2} \qquad (10)$$

defines the critical speed of the charge; if $\beta < \beta_o$, emission of ordinary waves becomes impossible.*

To calculate the integral in Eq. (8), it is convenient to introduce spherical coordinates, with the polar axis along the axis of the cone of normals:†

$$n_1 = r_1\cos\vartheta - r_2\sin\vartheta\cos\varphi,$$

$$n_2 = r_2\cos\vartheta + r_1\sin\vartheta\cos\varphi, \quad n_3 = \sin\vartheta\sin\varphi. \qquad (11)$$

We then get from Eq. (8)

$$S_{(o)}(\omega) = \frac{e^2\omega\mu}{2\pi c^2}\int_0^{2\pi} d\varphi\left[1-\left(\frac{\beta_o}{\beta}\right)^2\right]\frac{r_2^2\sin^2\varphi}{1-(r_1\beta_o/\beta - r_2\sqrt{1-(\beta_o/\beta)^2}\cos\varphi)^2}. \qquad (12)$$

If we write the denominator in (12) as the product of two polynomials linear in $\cos\varphi$, the whole expression becomes the sum of two elementary integrals, and by calculating them we get the expression

$$[2-(r_1+\beta_o/\beta) - |r_1-\beta_o/\beta|]\pi.$$

The sign of the difference $r_1 - \beta_o/\beta$ has an intuitive geometrical meaning, which follows from Eq. (9) and the meaning of the direction cosine r_1, namely: if $r_1 < \beta_o/\beta$, the optic axis is outside the cone of normals, and if $r_1 > \beta_o/\beta$, it is inside the cone. Thus for the cases of optic axis inside and outside the cone we have the respective formulas

$$S_{(o)}(\omega) = e^2\omega\mu c^{-2}(1-r_1), \qquad (13)$$

$$S_{(o)}(\omega) = e^2\omega\mu c^{-2}(1-\beta_o/\beta). \qquad (14)$$

Let us consider these expressions in more detail. If the charge moves along the optic axis ($r_1 = 1$), then (13) holds for all speeds larger than the critical, and $S_{(o)}(\omega) = 0$, as has already been shown by Ginzburg.[1] For other directions of motion of the charge, the emission increases continuously from zero for $\beta > \beta_o$, according to Eq. (14). Furthermore, if $r_1 > \beta_o$ there is a speed ($\beta = \beta_o/r_1$) at which Eq. (14) goes over continuously into Eq. (13), and for higher speeds of the charge there is no further increase of the ordinary radiation (per unit path length). If, finally, $r_1 < \beta_o$ (this means that the angle between the direction of motion of the charge and the optic axis is sufficiently large), then Eq. (14) is valid for all speeds larger than critical. In particular, for motion of the charge perpendicular to the optical axis, Eq. (14) agrees with the result previously obtained.[2]

Finally, let us examine the polarization of the ordinary waves. If the normal runs around the conical surface, then according to Eqs. (7) and (11) the positions of the vector $\mathbf{e}^{(o)}$ form a fan in the plane perpendicular to the optic axis. The axis of this fan is the perpendicular to the principal plane. The vector $\mathbf{e}^{(o)}$ reaches the extreme positions in the fan for the minimum value of $|e_3^{(o)}|$. If the optic axis is located outside the cone calculation shows that this minimum value occurs for vectors $\mathbf{e}^{(o)}$ that correspond to normals that satisfy the condition

$$r_1 - \mathbf{n}\mathbf{r}\, n_1 = 0.$$

It is clear that these normals are the lines in which the cone touches planes passing through the optic axis. The angle 2η between these tangent planes is the vertex angle of the fan of directions, and we have for it the value given by

$$\cos\eta = r_2^{-1}\sqrt{(\beta_o/\beta)^2 - r_1^2}. \qquad (15)$$

For a given speed of the charge the angle η takes its smallest value in the case of motion perpendicular to the optic axis, when it is equal to $\Theta_{(o)}$. If the direction of motion approaches the optic axis, the angle η increases, and when the cone touches the optic axis it reaches the value $\pi/2$. The directions of the polarization vector then fill an entire plane. It is clear that this is also true when the optic axis is inside the cone.

THE CONE OF NORMALS OF THE EXTRAORDINARY WAVES

In this section we are concerned with the geometrical properties of the emitted light waves. We shall return later to the question of the intensity of the radiation. For the unit vector $\mathbf{d}^{(e)}$ we get the following expression from (3) and (7):

*We use the indices o and e without parentheses to denote quantities associated with the critical speed.

†See Fig. 1, which illustrates the more general case of the extraordinary waves. In the present case the directions $\mathbf{r}$ and $\mathbf{k}$ are identical.

$$\mathbf{d}^{(e)} \equiv (1-n_1^2, -n_1 n_2, -n_1 n_3)/\sqrt{1-n_1^2}. \tag{16}$$

The vector $\mathbf{e}^{(e)}$ and the speed of propagation of the phase $v_{(e)}$ are determined from (3) and (5):

$$\mathbf{e}^{(e)} \equiv ((1-n_1^2)/\tau, -n_1 n_2/\varepsilon, -n_1 n_3/\varepsilon)/\sqrt{1-n_1^2}, \tag{17}$$

$$v_{(e)} = (c/\sqrt{\mu})\,[1/\tau + (1/\varepsilon - 1/\tau) n_1^2]^{1/2}. \tag{18}$$

The direction $\mathbf{s}^{(e)}$ of the extraordinary ray and the speed along the ray $u_{(e)}$ are found from (17) and (7):

$$\mathbf{s}^{(e)} \equiv (\tau n_1, \varepsilon n_2, \varepsilon n_3)/\sqrt{\varepsilon^2 + (\tau^2 - \varepsilon^2) n_1^2}, \quad u_{(e)} = v_{(e)}/\mathbf{n}\mathbf{s}^{(e)}. \tag{19}$$

Now we know all the quantities that determine the extraordinary waves, so that we can turn to the study of the geometrical shape of the cone of normals; its equation is determined by the argument of the δ function in Eq. (2):

$$r_1 n_1 + r_2 n_2 = v_{(e)}/c\beta. \tag{20}$$

Just as in the study of the ordinary waves, we introduce spherical coordinates, but postulate merely that the polar axis $\mathbf{k}$ is in the principal plane (Fig. 1). We get

$$n_1 = k_1 \cos\vartheta - k_2 \sin\vartheta \cos\varphi, \quad n_2 = k_2 \cos\vartheta + k_1 \sin\vartheta \cos\varphi,$$
$$n_3 = \sin\vartheta \sin\varphi. \tag{21}$$

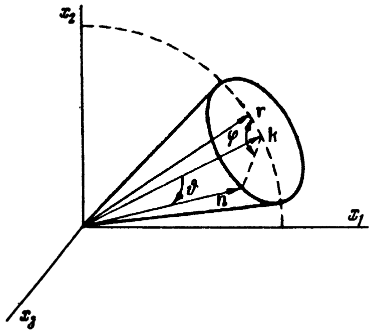

FIG. 1

Then the equation of the cone of normals takes the following form:

$$P\cos^2\vartheta + Q\sin^2\vartheta\cos^2\varphi + 2[A k_1 k_2 - r_1 r_2 (k_1^2 - k_2^2)]\sin\vartheta\cos\vartheta\cos\varphi = R. \tag{22}$$

As a simplification we introduce the following notations

$$A = (r_1^2 - 1/\varepsilon\mu\beta^2) - (r_2^2 - 1/\tau\mu\beta^2), \quad R = 1/\tau\mu\beta^2,$$
$$P = (k_1 r_1 + k_2 r_2)^2 - (1/\varepsilon - 1/\tau) k_1^2/\mu\beta^2,$$
$$Q = (k_2 r_1 - k_1 r_2)^2 - (1/\varepsilon - 1/\tau) k_2^2/\mu\beta^2. \tag{23}$$

We use the indeterminacy still remaining in the choice of the direction cosines k_1 and k_2 to simplify Eq. (22). We impose the condition

$$A k_1 k_2 - r_1 r_2 (k_1^2 - k_2^2) = 0, \tag{24}$$

which together with $k_1^2 + k_2^2 = 1$ determines k_1 and k_2:

$$k_1 = \frac{1}{\sqrt{2}}\left(1 + \frac{A}{(A^2 + 4r_1^2 r_2^2)^{1/2}}\right)^{1/2},$$
$$k_2 = \frac{1}{\sqrt{2}}\left(1 - \frac{A}{(A^2 + 4r_1^2 r_2^2)^{1/2}}\right)^{1/2}. \tag{25}$$

From Eqs. (22) and (24) we now get the following expression for the cone of the normals of the extraordinary waves:

$$\vartheta = \Theta_{(e)}(\varphi), \tag{26}$$

where we determine $\Theta_{(e)}$ from the equation

$$\cos\Theta_{(e)} = \left(\frac{R - Q\cos^2\varphi}{P - Q\cos^2\varphi}\right)^{1/2}. \tag{27}$$

From the right member of this last equation it can be seen that the cone has a symmetry axis, namely the polar axis of our spherical coordinates. In the general case the cross section of the cone will be an ellipse, as also follows from the geometrical interpretation given by Pafomov.[4,5]

The critical speed of the charge is fixed by the condition $\Theta_{(e)} = 0$. For this condition to hold, it is necessary that $P = R$, or, by Eq. (23),

$$(r_1^2 - 1/\varepsilon\mu\beta^2) k_1^2 + (r_2^2 - 1/\tau\mu\beta^2) k_2^2 + 2k_1 k_2 r_1 r_2 = 0. \tag{28}$$

From this equation and Eq. (25) we get for the critical speed

$$\beta_e = 1/\sqrt{(\varepsilon r_1^2 + \tau r_2^2)\mu} \tag{29}$$

In this case the cone of normals degenerates into a line, whose direction cosines are

$$k_{1e} = \varepsilon r_1/\sqrt{\varepsilon^2 r_1^2 + \tau^2 r_2^2}, \quad k_{2e} = \tau r_2/\sqrt{\varepsilon^2 r_1^2 + \tau^2 r_2^2}. \tag{30}$$

Finally, the speed of propagation of the phase of this critical wave is

$$v_e = \frac{c}{\sqrt{\mu}}\left(\frac{\varepsilon r_1^2 + \tau r_2^2}{\varepsilon^2 r_1^2 + \tau^2 r_2^2}\right)^{1/2}. \tag{31}$$

If we set in Eq. (19) $\mathbf{n} = \mathbf{k}_e$ and $v_{(e)} = v_e$, we get $\mathbf{s}^e = \mathbf{r}$ and $u_e = c\beta_e$. This result agrees with the assertion of Frank[5] that the first wave to appear is one with its ray directed along the motion of the charge. Under this condition the speed of the charge equals the ray velocity.

For particles with speeds close to the critical value, the expressions obtained for k_1, k_2, and $\cos\Theta_{(e)}$ can be simplified somewhat if we expand them in powers of $\delta = 1 - \beta_e/\beta$ and take the first two terms:

$$k_1 = k_{1e}[1 - 2k_{2e}^2\mu(\varepsilon - \tau)(v_e/c)^2\delta],$$
$$k_2 = k_{2e}[1 + 2k_{1e}^2\mu(\varepsilon - \tau)(v_e/c)^2\delta],$$
$$\cos\Theta_{(e)} = 1 - (\tau/\varepsilon)(v_e/v_o)^4[\cos^2\varphi + (v_e/v_o)^2\sin^2\varphi]^{-1}\delta. \quad (32)$$

It can be seen from (30) that in negative crystals ($\epsilon > \tau$) the direction of propagation of the wave that appears at the critical speed is between the direction of motion of the charge and the optic axis. In positive crystals ($\epsilon < \tau$) the normal lies outside the angle formed by the direction of motion of the charge and the optic axis. If the speed of the charge increases, then according to Eq. (32) the axis of the cone approaches the direction of motion of the charge in all crystals, but does not coincide with it even for $\beta = 1$ (except for the special cases $r_1 = 1$ and $r_1 = 0$, for which the axis of the cone and the direction of motion coincide).

The cones of the ordinary and extraordinary waves cannot intersect. They come in contract only for $\beta = \beta_0/r_1$, when they touch along the optic axis.

The polarization vectors $\mathbf{d}^{(e)}$ of the extraordinary waves are given by Eq. (16). If we use (21) and (26) to express $\mathbf{n}$, Eq. (16) gives parametric equations for the cone of polarizations. The analytic expression for the directrix of this cone is rather complicated. The use of the following estimate suffices for our further considerations.

If the optic axis is outside the cone of normals, the directions of polarization are contained in a pyramid whose faces are the following planes: 1) the tangent planes of the cone of normals that pass through the optic axis x_1; 2) planes through the axis x_3 and perpendicular to the generators of the cone of normals that lie in the principal plane (x_1, x_2) (see Fig. 2, which shows the case $r_1 = 0$. The pyramids must be continued symmetrically on the other half of the optic axis. This remark also applies to Fig. 3).

If the optic axis is inside the cone of normals, the picture changes, since there are no tangent planes passing through the optic axis. Figure 3 shows the cone of polarizations for $r_1 = 1$.

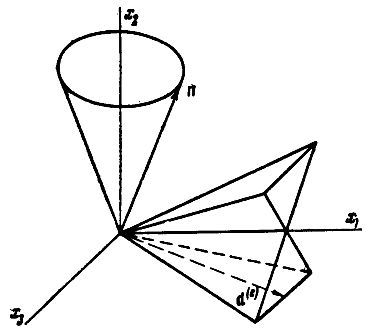

FIG. 2

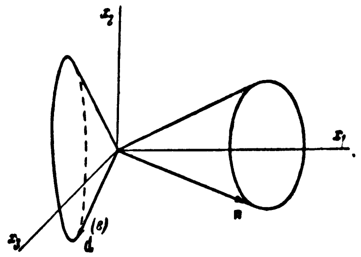

FIG. 3

THE ENERGY OF THE EXTRAORDINARY WAVES

Let us now turn to the calculation of the energy which the charge loses in the form of extraordinary waves. On the cone of normals the following equation holds:

$$\mathbf{r}\mathbf{e}^{(e)} = \frac{r_1\varepsilon\mu(v_{(e)}/c)^2 - \mathbf{nr}\, n_1}{\varepsilon\sqrt{1-n_1^2}} = \frac{\mathbf{nr}}{\varepsilon\sqrt{1-n_1^2}}\left(r_1\left(\frac{\beta}{\beta_o}\right)^2\mathbf{nr} - n_1\right).$$

If we also use the identity

$$\delta\left(\mathbf{nr} - \frac{v_{(e)}}{c\beta}\right) = \frac{\mathbf{nr}\,\delta(\cos\vartheta - \cos\Theta_{(e)})}{\sqrt{(R - Q\cos^2\varphi)(P - Q\cos^2\varphi)}},$$

then formula (2) takes the form

$$S_{(e)}(\omega) = \frac{e^2\omega\mu}{2\pi c^2}\left(\frac{\beta_o}{\beta}\right)^2\int_0^{2\pi}\frac{d\varphi}{\sqrt{(R - Q\cos^2\varphi)(P - Q\cos^2\varphi)}}$$
$$\times\left\{-\frac{\varepsilon}{\tau}\left(\frac{\beta_o}{\beta}\right)^2\frac{1}{\mathbf{nr}} + \frac{1}{1-n_1^2}\left[\mathbf{nr}\left(r_1\left(\frac{\beta}{\beta_o}\right)^2 - 1\right)\right.\right.$$
$$\left.\left. + 2n_2r_2\right]\right\}. \quad (33)$$

It will be shown in the Appendix that the value of this integral also depends essentially on the relative position of the cone of normals and the optic axis. Upon making the calculations we get (see Appendix) for the respective cases of optic axis inside and outside the cone:

$$S_{(e)}(\omega) = e^2\omega\mu c^{-2}(r_1 - \beta_o\beta_e/\beta^2), \quad (34)$$
$$S_{(e)}(\omega) = e^2\omega\mu c^{-2}(1 - \beta_e/\beta)\beta_o/\beta. \quad (35)$$

An examination of these formulas shows that as the speed of the charge increases beyond the critical value $c\beta_e$ the radiated energy increases continuously from zero in accordance with Eq. (35). In the region of directions of motion of the charge in which the equation $\beta = \beta_0/r_1$ can be satisfied, Eq. (35) goes over continuously into Eq. (34). For motion along the optic axis Eq. (34) holds for all speeds. In this case, and also for motion perpendicular to the optic axis, our formulas agree with results obtained previously.[2]

THE TOTAL RADIATION

Another matter of interest is the expression for the total energy radiated, $S(\omega) = S_{(o)}(\omega)$

$= S_{(e)}(\omega)$. At first, when the speed of the charge is between the critical speeds, waves of only one polarization appear. In this range of speeds the optic axis is outside the cone, and the total radiation is given either by

$$S(\omega) = e^2\omega\mu c^{-2}(1 - \beta_o/\beta), \tag{36}$$

or by

$$S(\omega) = e^2\omega\mu c^{-2}(1 - \beta_e/\beta)\,\beta_o/\beta, \tag{37}$$

depending on whether the crystal is negative or positive. For isotropic media this region is absent, since for such substances $\beta_o = \beta_e$.

For speeds at which both ordinary and extraordinary waves exist (and also always for $r_1 = 1$) the energy of the total radiation is given by the formula

$$S(\omega) = e^2\omega\mu c^{-2}(1 - \beta_o\beta_e/\beta^2), \tag{38}$$

which is independent of the relative positions of the optic axis and the cone of normals. The analogy between Eq. (38) and the well known formula for the radiation in an isotropic medium can be seen at a glance. It is curious that the formulas (36) — (38) for the total radiation involve the characteristic optical properties of the crystal and the parameters of the motion of the charge only through the quantities β_o/β and β_e/β.

CONCLUSION

The results obtained show that in spite of the complications of the structure of the electromagnetic field in an anisotropic medium, simple expressions can be given for the main properties of the Cerenkov radiation in uniaxial crystals. This is true particularly for the total radiated energy, which plays a part in a number of practical applications (counters, radiation losses in the motion of particles through matter).

For direct comparison with experiment one must take into account the influence of the surface of the crystal. The transition radiation which arises at the surface can be neglected, just as it is in the study of the radiation in isotropic media. The change of shape of the cones of normals on emergence from the crystal can be taken into account by using the law of refraction for the wave normals. The bounding surface of the crystal can affect the intensity of the radiation in a more complicated way.

In conclusion I take this opportunity to express my gratitude to Professor I. M. Frank for his interest and for helpful discussions.

APPENDIX

We write the formula (33) in the following way:

$$S_{(e)}(\omega) = (e^2\omega\mu/2\pi c^2)(\beta_o/\beta)^2(I_1 + I_2). \tag{1'}$$

The forms of the integrals I_1 and I_2 follow directly from Eq. (33). Expressing **nr** by using Eq. (21), rationalizing the denominator, and omitting terms odd in $\cos\varphi$, we get

$$I_1 = -\frac{\varepsilon}{\tau}\left(\frac{\beta_o}{\beta}\right)^2\int_0^{2\pi} d\varphi \frac{k_1r_1 + k_2r_2}{R\,(k_1r_1 + k_2r_2)^2\sin^2\varphi - T\cos^2\varphi},$$

$$T = (k_1r_1 + k_2r_2)^2(Q - R) + (k_2r_1 - k_1r_2)^2(P - R) = -(\tau\mu)^{-1}(\beta_o/\beta_e)^2.$$

The integration gives the result

$$I_1 = -2\pi\beta_e/\beta_o. \tag{2'}$$

In a similar way we get for I_2 the following expressions:

$$I_2 = 2\int_{-\infty}^{\infty}\frac{Uy^2 + x}{(Wy^2 - 2iDy - M)(Wy^2 + 2iDy - M)}dy$$

$$= 4\int_{-\infty}^{\infty}\frac{ay + b}{Wy^2 - 2iDy - M}dy, \tag{3'}$$

where

$W = P - k_1^2R,\quad D = k_1(P - R),$

$M = k_1^2(P - R) + k_2^2(Q - R) = r_1^2 - (\beta_o/\beta)^2,$

$U = W\,[(\beta/\beta_o)^2(k_1r_1 + k_2r_2)M + 2r_2k_2],$

$x = (\beta/\beta_o)^2M\,[2r_1D - (\beta/\beta_o)^2(k_1r_1 + k_2r_2)M - 2k_2r_2(\beta_o/\beta)^2],$

$a = (UM + xW)/4MDi, \qquad b = -x/2M.$

After integration we get

$$I_2 = \frac{4\pi i}{W}\frac{1}{z_2 - z_1}[(ib - az_1)\operatorname{sign} z_1 - (ib - az_2)\operatorname{sign} z_2],$$

where iz_1 and iz_2 are the roots of the denominator in Eq. (3′):

$$z_{1,2} = W^{-1}(D \pm k_2r_2\beta_o/\beta).$$

The root z_1 is positive. The sign of z_2 depends on the relative position of the optic axis and the cone of normals, since $z_2 = 0$ for $\beta = \beta_o/r_1$. We have finally

$$I_2 = 2\pi r_1(\beta/\beta_o)^2, \quad I_2 = 2\pi\beta/\beta_o, \tag{4'}$$

for $r_1 > \beta_o/\beta$ and $r_1 < \beta_o/\beta$, respectively. The formulas (34) and (35) follow directly from Eqs. (1′), (2′), and (4′).

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