FINITE-AMPLITUDE WAVES IN A MULTI-COMPONENT CONDUCTING MEDIUM

V. S. TKALICH

Submitted to JETP editor October 22, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 73-77 (July, 1960)

The set of equations for a non-ideal plasma in the hydrodynamic approximation is reduced to a linear one without assuming that the signal is small. Propagation of finite amplitude waves in the presence of a magnetic field is investigated; a comparison is made with well known results obtained on the basis of other assumptions.

1. The set of equations of the hydrodynamic approximation for a plasma consisting of N kinds of ions (each of which is treated as an incompressible fluid) has the following form[1-7]

$$\operatorname{curl}\mathbf{e} = -\frac{\mu}{c}\frac{\partial\mathbf{h}}{\partial t}, \qquad \operatorname{curl}\mathbf{h} = \frac{\varepsilon}{c}\frac{\partial\mathbf{e}}{\partial t} + \frac{4\pi}{c}\sum_{l=1}^{N} e_l n_l \mathbf{v}_l,$$

$$\operatorname{div}\mathbf{h} = 0, \qquad \operatorname{div}\varepsilon\mathbf{e} = 4\pi\sum_{l=1}^{N} e_l n_l, \qquad \operatorname{div}\mathbf{v}_k = 0,$$

$$\partial\mathbf{v}_k/\partial t + \nabla w_k = [\mathbf{v}_k\times[\operatorname{curl}\mathbf{v}_k + (\mu e_k/cm_k)\,\mathbf{h}]] + (e_k/m_k)\,\mathbf{e}$$

$$+\sum_{l=1}^{N}\nu_{kl}(\mathbf{v}_l - \mathbf{v}_k) - \nu_k \operatorname{curl}\operatorname{curl}\mathbf{v}_k$$

$$(k,\ l = 1, 2, \ldots, N), \tag{1}$$

where $w_k \equiv v_k^2/2 + p_k/m_k n_k + F_k$ is the sum of the kinetic, thermal and potential energy per unit mass of ions of the k-th kind; ν_k is the kinematic coefficient of viscosity; ν_{kl} is the effective frequency of collisions of an ion of type k with ions of type l.

We resolve the field π consisting of the set of quantities ($\mathbf{h}$, $\mathbf{e}$, v_k, w_k) into a field $\Pi_0(\mathbf{H}_0, \mathbf{E}_0, \mathbf{V}_{k0}, W_{k0})$ which represents some known solution of the system (1), and an auxiliary field to be determined $\Pi(\mathbf{H}, \mathbf{E}, \mathbf{V}_k, W_k)$

$$\mathbf{h} = \mathbf{H}_0 + \mathbf{H}, \qquad \mathbf{e} = \mathbf{E}_0 + \mathbf{E},$$

$$\mathbf{v}_k = \mathbf{V}_{k0} + \mathbf{V}_k, \qquad w_k = W_{k0} + W_k. \tag{2}$$

On introducing the vector potential $\mathbf{A}$ for the auxiliary field Π

$$\mathbf{H} = \operatorname{curl}\mathbf{A}, \qquad \mathbf{E} = -c^{-1}\,\partial\mathbf{A}/\partial t, \tag{3}$$

the system of equations for the field $\Pi(\mathbf{A}, \mathbf{V}_k, W_k)$ may be re-written in the form

$$\operatorname{div}\mathbf{A} = 0, \qquad \operatorname{div}\mathbf{V}_k = 0,$$

$$\left(\operatorname{curl}\operatorname{curl} + \frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = \frac{4\pi}{c}\sum_{l=1}^{N} e_l n_l \mathbf{V}_l, \tag{4}$$

$$\frac{\partial}{\partial t}\left(\mathbf{V}_k + \frac{\mu e_k}{cm_k}\mathbf{A}\right) + \nabla W_k = \left[\mathbf{V}_k\times\operatorname{curl}\left(\mathbf{V}_k + \frac{\mu e_k}{cm_k}\mathbf{A}\right)\right]$$

$$+\sum_{l=1}^{N}\nu_{kl}(\mathbf{V}_l - \mathbf{V}_k) - \nu_k\operatorname{curl}\operatorname{curl}\mathbf{V}_k + \left[\mathbf{V}_k\times\left(\operatorname{curl}\mathbf{V}_{k0}\right.\right.$$

$$\left.\left.+\frac{\mu e_k}{cm_k}\mathbf{H}_0\right)\right] + \left[\mathbf{V}_{k0}\times\operatorname{curl}\left(\mathbf{V}_k + \frac{\mu e_k}{cm_k}\mathbf{A}\right)\right]. \tag{5}$$

Equation (5) can be made linear if we restrict ourselves to the consideration of "helical" motions

$$\operatorname{curl}[\mathbf{V}_k + (\mu e_k/cm_k)\,\mathbf{A}] = a_k\mathbf{V}_k, \tag{6}$$

where $a_k = a_k(\mathbf{r}, t)$ satisfies the condition $(\mathbf{V}_k\nabla)\,a_k = 0$. The method of linearizing helical fields in ordinary hydrodynamics,[8,9] applied[10] to stationary helical motions of the form (6) is generalized in this paper to non-stationary fields. On setting $a_k = a_k(t)$ ("homogeneous" helical motion) and on substituting (6) into the equation of motion (5), we obtain

$$\frac{\partial}{\partial t}\left(\mathbf{V}_k + \frac{\mu e_k}{cm_k}\mathbf{A}\right) + \nabla W_k = \sum_{l=1}^{N}\nu_{kl}(\mathbf{V}_l - \mathbf{V}_k)$$

$$-\nu_k\operatorname{curl}\operatorname{curl}\mathbf{V}_k + \left[\mathbf{V}_k\times\left((\operatorname{curl} - a_k)\,\mathbf{V}_{k0} + \frac{\mu e_k}{cm_k}\mathbf{H}_0\right)\right]. \tag{7}$$

On replacing (7) by the condition that the equations are soluble (by taking the curl) and on taking (6) into account we obtain

$$\frac{\partial}{\partial t}(a_k\mathbf{V}_k) = \sum_{l=1}^{N}\nu_{kl}\operatorname{curl}(\mathbf{V}_l - \mathbf{V}_k) - \nu_k(\operatorname{curl})^3\,\mathbf{V}_k$$

$$+\left\{\left[(\operatorname{curl} - a_k)\,\mathbf{V}_{k0} + \frac{\mu e_k}{cm_k}\mathbf{H}_0\right]\nabla\right\}\mathbf{V}_k$$

$$-(\mathbf{V}_k\nabla)\left[(\operatorname{curl} - a_k)\,\mathbf{V}_{k0} + \frac{\mu e_k}{cm_k}\mathbf{H}_0\right]. \tag{8}$$

In the simplest case when the quantity in the square brackets does not depend on the coordinates, i.e., when

$$(\operatorname{curl} - a_k)\,\mathbf{V}_{k0} + (\mu e_k/cm_k)\,\mathbf{H}_0 = \mathbf{f}_k(t),$$

equation (8) assumes a particularly simple form

$$\frac{\partial}{\partial t}(a_k\mathbf{V}_k) = \sum_{l=1}^{N} \nu_{kl}\,\mathrm{curl}(\mathbf{V}_l-\mathbf{V}_k) - \nu_k(\mathrm{curl})^3\mathbf{V}_k + \{[(\mathrm{curl}-a_k)\mathbf{V}_{k0} + (\mu e_k/cm_k)\,\mathbf{H}_0]\,\nabla\}\,\mathbf{V}_k. \quad (9)$$

Thus, when the set Π_0 — a certain solution of the system (1) — is known, then (4), (6), and (8) represent a system of linear partial differential equations. Since this system is linear, the vector fields $\Pi(\mathbf{A}, \mathbf{V}_k)$ satisfy the superposition principle. We note that the problem of the validity of the superposition principle for finite amplitude fields has been previously studied in a number of papers.[6-12]

2. By utilizing the method developed for stationary helical motion[10] we obtain an equation for the vector potential $\mathbf{A}$, and also expressions for the partial currents. On introducing the notation

$$\Omega_k^2 \equiv 4\pi e_k^2 n_k/m_k, \qquad \Omega^2 \equiv \sum_{l=1}^{N}\Omega_l^2, \quad \mathbf{U}_k \equiv 4\pi e_k n_k c^{-1}\mathbf{V}_k,$$

$$\hat{\alpha}_k \equiv a_k - \mathrm{curl},\ \hat{\alpha}_0 \equiv -\mathrm{curl},\ \hat{\beta} \equiv \mathrm{curl\ curl} + \varepsilon\mu c^{-2}\partial^2/\partial t^2, \quad (10)$$

where all the operators $\hat{\alpha}_k$ and $\hat{\alpha}_0$ commute[13,14] with each other (and also with $\hat{\beta}$, if $a_k = \mathrm{const}$), we rewrite the system (4) and (6) in the following form

$$\sum_{l=1}^{N}\mathbf{U}_l = \hat{\beta}\mathbf{A}, \quad \hat{\alpha}_k\mathbf{U}_k + (\Omega_k/c)^2\hat{\alpha}_0\mathbf{A} = 0,$$
$$\mathrm{div}\,\mathbf{A} = 0, \quad \mathrm{div}\,\mathbf{U}_k = 0. \quad (11)$$

On multiplying the first equation of (11) by the operator $\hat{L}$

$$\hat{L} \equiv \prod_l \hat{\alpha}_l \equiv \hat{L}_k\hat{\alpha}_k, \quad (12)$$

where the product is taken over all the different $\hat{\alpha}_k$, and on utilizing the second equation of (11), we obtain the equation for the vector potential $\mathbf{A}$

$$\Big[\hat{L}\hat{\beta} + \hat{\alpha}_0\sum_{l=1}^{N}(\Omega_l/c)^2\hat{L}_l\Big]\mathbf{A} = 0. \quad (13)$$

On multiplying the second equation of (11) by a_k^{n-1} and on summing over all k we obtain the recurrence relation[10]

$$\sum_{l=1}^{N} a_l^n\mathbf{U}_l = \mathrm{curl}\Big[\sum_{l=1}^{N} a_l^{n-1}\mathbf{U}_l + \mathbf{A}\sum_{l=1}^{N} a_l^{n-1}(\Omega_l/c)^2\Big],$$

from which with the aid of the first of relations (11) we obtain

$$\sum_{l=1}^{N} a_l^n\mathbf{U}_l = \Big[(\mathrm{curl})^n\hat{\beta} + \sum_{l=1}^{N}(\Omega_l/c)^2\sum_{m=1}^{n} a_l^{n-m}(\mathrm{curl})^m\Big]\mathbf{A}. \quad (14)$$

On setting in (14) n = 1, 2, . . . , N − 1 we obtain together with the first equation of (11) a system of algebraic equations linear in $\mathbf{U}_k$. It is soluble if among the a_k there are no equal ones ($a_k \neq a_l$, $k \neq l$), since its determinant is a Vandermonde determinant.[14] On solving it we obtain a linear differential operator $\hat{\gamma}_k$ which on being applied to the vector potential $\mathbf{A}$ yields

$$\mathbf{U}_k = \hat{\gamma}_k\mathbf{A}. \quad (15)$$

3. We consider fields in a neutral plasma in the presence of a constant homogeneous magnetic field $\mathbf{H}_0$ = const. On setting

$$\mathbf{E}_0 = \mathbf{V}_{k0} = \nu_{kl} = \nu_k = 0, \qquad a_k = \mathrm{const},$$
$$W_{k0} = \mathrm{const}, \qquad \varepsilon = \mu = 1,$$

we obtain from (9) with the aid of (15) the following equation

$$\hat{\gamma}_k[a_k\partial/\partial t - (e_k/cm_k)(\mathbf{H}_0\nabla)]\,\mathbf{A} = 0, \quad (16)$$

which must be solved simultaneously with (13).

For a progressive wave of the simple harmonic form

$$\mathbf{A} = \mathbf{A}_0\exp\{i(\omega t - \varkappa\mathbf{nr})\}, \qquad \mathbf{V}_k = \mathbf{V}_{0k}\exp\{i(\omega t - \varkappa\mathbf{nr})\} \quad (17)$$

(ω is the frequency, κ is the wave number, $\mathbf{n}$ is a unit vector in the direction of motion of the wave, $\mathbf{A}_0$ and $\mathbf{V}_{0k}$ are complex amplitudes) on substituting (17) into (16) we obtain

$$a_k = -\varkappa\mathbf{n}\omega_k/\omega, \qquad \omega_k \equiv e_k\mathbf{H}_0/cm_k. \quad (18)$$

On substituting (17) into (6) we obtain the amplitudes

$$\mathbf{V}_{0k} = (e_k/cm_k)(s\mathbf{n}\omega_k/\omega - 1)^{-1}\mathbf{A}_0, \quad \mathbf{A}_0 = [\mathbf{n}\times\mathbf{b}] + is[\mathbf{n}\times[\mathbf{n}\times\mathbf{b}]], \quad (19)$$

where $\mathbf{b}$ is an arbitrary constant vector, the quantity s characterizes the polarization of the wave $s = \pm 1$. The vectors $\mathbf{A}$ and $\mathbf{V}_k$ whose amplitudes satisfy a relation of the form $\mathbf{A}_0\mathbf{A}_0 = 0$ are circularly polarized.

On utilizing (4), (17), and (19) we obtain in the usual manner[1,2] the dielectric constant of the plasma ϵ_0

$$\varepsilon_0 = 1 - \sum_{l=1}^{N}(\Omega_l/\omega)^2/[1 - (s\mathbf{n}\omega_l/\omega)], \quad (20)$$

and from the condition that the equations are soluble we obtain the phase velocity $V_{ph} \equiv \omega/\kappa$

$$V_{ph} = \pm c/\sqrt{\varepsilon_0}. \quad (21)$$

It follows from (20) and (21) that there exist two waves (corresponding to $s = \pm 1$) of different polarization (leading to different ϵ_0) which are propagated with different phase velocities.

If $\omega \to s\mathbf{n}\omega_k$, then for a fixed $\mathbf{A}_0$ the velocity $\mathbf{V}_{0k}$ should in accordance with (19) increase without limit. However, the relation (19) and the approximation itself cease to hold when V_{0k} becomes comparable to the corresponding thermal velocity V_{kT}.

4. We now give the principal conditions under which the hydrodynamic approximation is applic-

able to plasma. Let L and T be the characteristic length and time for the process; let λ_k and τ_k be the mean free path and the corresponding time, and let V_{kT} be the thermal velocity for the k-th type of ions. We can consider the medium to be continuous if the process is sufficiently slow ($T \gg \tau_k$) and the gas is sufficiently dense ($L \gg \lambda_k$). The pressure p_k is scalar if the magnetic fields are sufficiently weak ($2\pi c m_k/He_k \gg \tau_k, \lambda_k/V_{kT}$).

The plasma may be regarded as incompressible if its temperature is sufficiently great, i.e., if the velocity of thermal motion V_{kT} is considerably greater than both the velocity of ordered motion v_k ($V_{kT} \gg |v_k|$), and also the velocity of propagation of the disturbance being considered in the given approximation ($V_{kT} \gg V_{ph}$).

If for a certain type of motion one of the kinds of ions (there may be several such kinds) plays no essential role (for example, the partial currents due to it are small), then the above conditions need not necessarily be satisfied for it. The latter conditions are similar to the conditions for the applicability of ordinary magnetohydrodynamics,[4,12,15] but are not so restrictive.

We assume that the plasma has two components. Then with the aid of (20) and (21) we obtain the expression for the phase velocity

$$V_{ph}^2 = (c\omega / \Omega)^2 (1 - s\mathbf{n}\omega_l/\omega)(s\mathbf{n}\omega_i/\omega - 1), \qquad (22)$$

which coincides with the expression obtained from formula (13) of Braginskiĭ's paper[3] if the plasma is regarded as incompressible, and if the transition is made to the limit c_l, $c_i \to \infty$.

On assuming that the frequencies in formula (22) are small ($\mathbf{n}\omega_i \gg \omega$) we obtain an expression for the phase velocity

$$V_{ph} \approx (H_0/\sqrt{4\pi M n}) \cos(\widehat{\mathbf{nH}_0}),$$

which coincides respectively with expressions (2.23) and (2.18) of Syrovat-skiĭ's paper[15] for the Alfven and for the slow magnetoacoustic waves (in the limiting case of high temperature $V_T \to \infty$).

We note that the well known expressions[1,2] for the dielectric permittivity tensor were obtained for the case of zero temperature ($V_{kT} \to 0$) and for infrequent collisions ($T \ll \tau_k$), i.e., in a limiting case which is the opposite to the one considered here.

5. The boundary conditions which have to be adopted in solving the system (1) assume different forms depending on the properties of the plasma boundary and on the nature of the process. If an ideal ($\nu_{kl} = \nu_k = 0$) plasma is bounded by a dielectric, then on integrating (1) in the simplest case when there are no surface charges or currents we obtain the following boundary conditions:[5]

$$[\mathbf{h}_t] = [\mathbf{e}_t] = [\mu \mathbf{h}_n] = [\varepsilon \mathbf{e}_n] = \mathbf{v}_{kn} = 0. \qquad (23)$$

The square brackets denote the difference between the values of the quantity contained within the brackets taken on the two sides of the boundary; the index t denotes the tangential (to the plasma boundary) components of the vectors, while the index n denotes the normal components. On utilizing (2) we bring the conditions (23) into the form

$$[(\mathbf{H}_0 + \mathbf{H})_t] = [(\mathbf{E}_0 + \mathbf{E})_t] = [\mu(\mathbf{H}_0 + \mathbf{H})_n]$$

$$= [\varepsilon(\mathbf{E}_0 + \mathbf{E})_n] = (\mathbf{V}_{k0} + \mathbf{V}_k)_n = 0,$$

which is the most convenient one for the solution of problems by means of the present method.

In conclusion the author expresses his deep gratitude to N. V. Saltanov and E. F. Tkalich for discussions.

[1] E. Aström, Ark. f. Fys. **2**, 443 (1951).

[2] Al'pert, Ginzburg, and Feĭnberg, Распространение радиоволн (Propagation of Radio Waves), Gostekhizdat, (1953).

[3] S. I. Braginskiĭ, Dokl. Akad. Nauk SSSR, **115**, 473 (1957).

[4] S. I. Braginskiĭ, Collection Вопросы магнитной гидродинамики и динамики плазмы (Problems of Magnetohydrodynamics and Plasma Dynamics), Riga, 1959, p. 67.

[5] Chandrasekhar, Kaufman, and Watson, Ann. Phys. **5**, 1 (1958).

[6] V. S. Tkalich, loc. cit. ref. 4, p. 191.

[7] V. S. Tkalich, Izv. Akad. Nauk SSSR Ser. Fiz., OTN, MM, No 5, (1959).

[8] I. S. Gromeka, Collected Works, USSR Acad. Sci., 1952, p. 76.

[9] O. F. Vasil'ev, Основы механики винтовых и циркуляционных потоков (Fundamentals of the Mechanics of Helical and Circulating Flows), Gosenergoizdat, (1958).

[10] E. F. Tkalich and V. S. Tkalich, Izv. Akad. Nauk SSSR Ser. Fiz., OTN, MM, in press.

[11] J. N. Kapur, Appl. Sci. Research 8A, 2-3, 198 (1959).

[12] H. Alfen, Cosmical Electrodynamics, Oxford, (1950).

[13] N. E. Kochin, Векторное исчисление и начала тензорного исчисления (Vector Calculus and Intro-

duction to Tensor Calculus), USSR Acad. Sci., (1951).

[14] V. I. Smirnov, Курс высшей математики (Course of Higher Mathematics), Vol. 3, part 1, Gostekhizdat, (1953).

[15] S. I. Syrovat-skiĭ, Usp. Fiz. Nauk, 62, 247 (1957).

Translated by G. Volkoff
15