SOLUTION OF EQUATION FOR THE MESON-MESON SCATTERING AMPLITUDE IN THE ASYMPTOTIC REGION

A. A. ANSEL'M

Leningrad Physico-Technical Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor February 2, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) 38, 1887-1890 (June, 1960)

The set of integral equations deduced in references 1 and 2 for the meson-meson scattering amplitude is solved exactly. An integral equation for the amplitude which depends only on one variable is obtained as a direct mathematical corollary of this set.

THE asymptote of the meson-meson scattering amplitude was studied in references 1 and 2 by means of the summation of the "parquet" sequence of Feynman diagrams. A set of integral equations was obtained which determine the amplitude as a function of two variables (actually, the transferred momentum and the energy) and, in independent fashion, the equation for the amplitude which depends on a single variable, corresponding to the case in which all the invariants of the problem are quantities of the same order. However, it was not possible for the authors to obtain the latter equation as a direct mathematical corollary of their system of equations; moreover, the given system was solved only in certain limiting cases. In the present work, the equation for the amplitude which depends on a single variable was derived directly from the set of integral equations, and an exact solution of this system was obtained. In limiting cases the resultant solution coincides with the solution given in references 1 and 2, if we correct the computational error made in these papers.

As was shown in references 1 and 2, the amplitude P in the case of neutral pseudoscalar theory consists of three terms which can be conveniently chosen as functions of the variables x and y:

$$P(x, y) = R_0(x) + 2\Phi(x) + \Phi(x, y), \quad R_0(x) = 24(1-x),$$
$$\Phi(x) = \Phi(x, x), \quad x = \left[1 + \frac{5g_0^2}{4\pi}\ln\frac{\Lambda^2}{t}\right]^{-1/5},$$
$$y = \left[1 + \frac{5g_0^2}{4\pi}\ln\frac{\Lambda^2}{s}\right]^{-1/5}. \tag{1}$$

Here Λ^2, s, and t are squares of the cutoff momentum, energy, and transferred momentum, and g_0 is the unrenormalized coupling constant. The quantity $R_0(x)$ is the contribution to the amplitude of the simplest diagram, while $\Phi(x)$ and $\Phi(x, y)$ are the contributions of the most important ("contractible") diagrams. The quantity $P(x, y)$ satisfies the integral equation

$$P(x, y) = s(x) - \frac{1}{2}\int_y^x s(x) P(t, y)\frac{dt}{t^2} - \frac{1}{2}\int_x^1 s(t) P(t, y)\frac{dt}{t^2},$$
$$s(x) = R_0(x) + 2\Phi(x) = P(x) - \Phi(x), \quad P(x) = P(x, x). \tag{2}$$

It is most important for the following that the variables in (2) are separable. In fact, if we set $P(x, y) = P_1(x)P_2(y)$, divide both parts of (2) by $P_2(y)$, and differentiate with respect to y, we easily obtain

$$P_2'(y)/P_2^2(y) = P_1(y)/2y^2, \tag{3}$$

so that x-dependent terms drop out of the equation for $P_2(y)$. It follows from (3) that

$$\frac{1}{2}\int_y^x \frac{P_1(t)}{t^2}dt = \frac{1}{P_2(y)} - \frac{1}{P_2(x)}. \tag{4}$$

The same integral enters into (2) also, upon substituting $P(t, y) = P_1(t)P_2(y)$. If, in accord with Eq. (4), we replace it by the difference $P_2^{-1}(y) - P_2^{-1}(x)$, an integral equation is obtained which does not contain y, and which, together with (4), comprises a system equivalent to (2). The relations that follow can be obtained by starting out from this pair of equations, but we shall set them up in a somewhat simpler fashion. Differentiating (2) with respect to x and setting x = y, we get

$$P_1'(x)P_2(x) = s'(x). \tag{5}$$

It follows from (3) and (5) that

$$P'(x) = s'(x) + P^2(x)/2x^2. \tag{6}$$

It is easy to see from the definitions (1) and (2) that $s(x) = \frac{1}{3}R_0(x) + \frac{2}{3}P(x)$, which gives the following final integral equation for an amplitude dependent on one variable:

$$P(x) = R_0(x) + \frac{3}{2}\int_1^x \frac{P^2(t)}{t^2}\,dt. \tag{7}$$

This result coincides with the equation for P (x) obtained in references 1 and 2 by a special consideration of the graphs. The solution of (7) is found in references 1 and 2 in the form

$$P(x) = \frac{\alpha+1}{3}\,x\,(1-x^{\alpha})\left[1+\frac{\alpha+1}{\alpha-1}x^{\alpha}\right]^{-1}, \qquad \alpha = \sqrt{145}. \tag{8}$$

The fact that P (x, y) = $P_1(x)P_2(y)$ makes it possible to find P (x, y). It is easy to see from (3) that

$$P_2(y) = \exp\left[\frac{1}{2}\int^y \frac{P(t)}{t^2}\,dt\right] = \mathrm{const}\cdot y^{(\alpha+1)/6}\left(1+\frac{\alpha+1}{\alpha-1}y^{\alpha}\right)^{-1/2}. \tag{9}$$

Hence:

$$P(x, y) = \frac{\alpha+1}{3}\,x\left(\frac{y}{x}\right)^{(\alpha+1)/6}(1-x^{\alpha})$$
$$\times\left[1+\frac{\alpha+1}{\alpha-1}x^{\alpha}\right]^{-3/2}\left[1+\frac{\alpha+1}{\alpha-1}y^{\alpha}\right]^{-1/2}. \tag{10}$$

The situation is somewhat more complicated in the case of symmetric pseudoscalar theory. Here the following relations hold by reason of the presence of isotopic variables[1,2]

$$P_{\xi_1\xi_2\xi_3\xi_4}(x, y) = P(x, y)\,\delta_s + P_1(x, y)\,\delta_{\xi_1\xi_2}\delta_{\xi_3\xi_4},$$
$$\Phi_{\xi_1\xi_2\xi_3\xi_4}(x, y) = \Phi(x, y)\,\delta_s + \Phi_1(x, y)\,\delta_{\xi_1\xi_2}\delta_{\xi_3\xi_4},$$
$$P(x, y) = \rho_0(x) + 2\Phi(x) + \Phi(x, y) + \Phi_1(x),$$
$$P_1(x, y) = \Phi_1(x, y) - \Phi_1(x),$$
$$\rho_0(x) = \frac{16}{3}(1-x), \quad x = \left[1+\frac{5g_0^2}{4\pi}\ln\frac{\Lambda^2}{t}\right]^{3/5},$$
$$y = \left[1+\frac{5g_0^2}{4\pi}\ln\frac{\Lambda^2}{s}\right]^{3/5}, \qquad \delta_s = \delta_{\xi_1\xi_2}\delta_{\xi_3\xi_4} + \delta_{\xi_1\xi_3}\delta_{\xi_2\xi_4} + \delta_{\xi_1\xi_4}\delta_{\xi_2\xi_3}, \tag{11}$$

ξ_i is the isotopic variable of the i-th meson. If we introduce the notation

$$L(s(x), P(x, y)) \equiv \frac{1}{3}\int_y^x s(x)\,P(t, y)\,\frac{dt}{t^2}$$
$$+\frac{1}{3}\int_x^1 s(t)\,P(t, y)\,\frac{dt}{t^2}, \tag{12}$$

then the corresponding set of integral equations can be written in the form

$$P(x, y) = s(x) + L(s(x), P(x, y)),$$
$$P_1(x, y) = -\Phi_1(x) + L[\tfrac{5}{2}(s(x) - \Phi_1(x)),$$
$$P(x, y) + P_1(x, y)] + L(\Phi_1(x), P_1(x, y)),$$
$$s(x) = \rho_0(x) + 2\Phi(x) + \Phi_1(x)$$
$$= \tfrac{2}{3}P(x) + \tfrac{1}{3}\rho_0(x) + \tfrac{1}{3}\Phi_1(x). \tag{13}$$

The first of these equations has the same structure as Eq. (2). Again setting P(x, y) = $P_1(x)P_2(y)$, and repeating the calculations, we obtain from the first equation of (13)

$$\frac{P_2'(y)}{P_2(y)} = -\frac{P(y)}{3y^2}, \quad P_1'(x)\,P_2(x) = s'(x),$$
$$P(x) = s(x) - \frac{1}{3}\int_1^x \frac{P^2(t)}{t^2}\,dt. \tag{14}$$

Or, taking into account Eq. (13) for s (x),

$$P(x) = \rho_0(x) + \Phi_1(x) - \int_1^x \frac{P^2(t)}{t^2}\,dt. \tag{15}$$

Using the first equation of (13), we can transform the second equation into a form coinciding with (2):

$$G(x, y) = \lambda(x) + L\,[\tfrac{3}{2}\lambda(x), G(x, y)],$$
$$G(x, y) = \tfrac{5}{3}P(x, y) + P_1(x, y), \qquad \lambda(x) = \tfrac{5}{3}s(x) - \Phi_1(x). \tag{16}$$

It is then clear that G(x, y) = $G_1(x)G_2(y)$, whence

$$\frac{G_2'(y)}{G_2(y)} = -\frac{G(y)}{2y^2}, \quad G_1'(x)\,G_2(x) = \lambda'(x),$$
$$G(x) = \lambda(x) - \frac{1}{2}\int_1^x \frac{G^2(t)}{t^2}\,dt. \tag{17}$$

Since $P_1(x, x) = 0$, then, in accord with (16), G (x) differs from P (x) by only the factor 5/3. Eliminating $\Phi_1(x)$ from (17) and (15), we easily obtain

$$P(x) = \rho_0(x) - \frac{11}{6}\int_1^x \frac{P^2(t)}{t^2}\,dt, \tag{18}$$

which again is identical with the result of references 1 and 2. The solution of this equation is

$$P(x) = \tfrac{16}{11}x\,(1 - x^{-19/3})/(1 + \tfrac{8}{11}x^{-19/3}), \tag{19}$$

whence, by the method given above, we can easily obtain P (x, y):

$$P(x, y) = \tfrac{16}{11}x\,(x/y)^{16/33}(1 - x^{-19/3})$$
$$\times(1 + \tfrac{8}{11}x^{-19/3})^{-9/11}(1 + \tfrac{8}{11}y^{-19/3})^{-3/11}. \tag{20}$$

In order to find $P_1(x, y)$, we can use the relation (17) and the first equation of (14). Since G (x) = 5/3 P (x), we obtain

$$G_2'/G_2 = 5P_2'/2P_2,$$
$$\text{i. e. } G_2 = P_2^{5/2}, \qquad G_1 = \tfrac{5}{3}P_1P_2^{-3/2}, \tag{21}$$

whence

$$P_1(x, y) = \tfrac{5}{3}(\tfrac{16}{11}x)\,x^{16/33}y^{-16/33}(1 - x^{-19/3})(1 + \tfrac{8}{11}x^{-19/3})^{-9/11}$$
$$\times(1 + \tfrac{8}{11}y^{-19/3})^{-3/11}[(x/y)^{8/11}(1 + \tfrac{8}{11}x^{-19/3})^{3/11}$$
$$\times(1 + \tfrac{8}{11}y^{-19/3})^{-3/11} - 1]. \tag{22}$$

In references 1 and 2, P (x, y) and $P_1(x, y)$ were obtained in the limiting cases $x - 1 \ll 1$,

$y - 1 \ll 1$ (perturbation theory) and $x \gg 1$, $y \gg 1$. If we correct the computational error contained in these papers, then the corresponding equations have the form

$$P(x, y) = \begin{cases} \frac{16}{3}(x-1)\{1 - \frac{88}{27}(x-1)^2 - \frac{8}{9}[(y-1)^2 - (x-1)^2] + \cdots\}, & x-1 \ll 1,\ y-1 \ll 1 \\ \left(\frac{16}{11}x\right) x^{16/33} y^{-16/33} + \cdots, & x \gg 1, \quad y \gg 1, \end{cases}$$

$$P_1(x, y) = \begin{cases} -\frac{5}{12}\left(\frac{16}{3}\right)^2 (x-1)[(y-1)^2 - (x-1)^2] + \cdots & x-1 \ll 1,\ y-1 \ll 1 \\ \left(\frac{16}{11}x\right)\frac{5}{3}\left(x^{40/33} y^{-40/33} - x^{16/33} y^{-16/33}\right) + \cdots, & x \gg 1, \quad y \gg 1, \end{cases}$$

which is identical to the expansion obtained directly from (20) and (22).

The most curious fact is that the scattering amplitude in the asymptotic region is a product of functions each of which depends only on the transferred momentum and the energy. It is not clear, however, to what degree this property is preserved if we do not limit ourselves to a consideration of "parquet" diagrams only.

The author is grateful to V. V. Anisovich for a number of useful remarks.

[1] Dyatlov, Sudakov, and Ter-Martirosyan, JETP **32**, 767 (1957), Soviet Phys. JETP **5**, 631 (1957).

[2] Pomeranchuk, Sudakov, and Ter-Martirosyan, Phys. Rev. **103**, 784 (1956).

Translated by R. T. Beyer
359