CRITICAL VELOCITY FOR RADIATION OF LIGHT IN OPTICALLY ANISOTROPIC MEDIA

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The radiation of an optical source characterized by an arbitrary frequency which moves uniformly in an optically anisotropic transparent medium is considered. The phase velocity, the velocity along the ray, and the group velocity of the radiation produced by such a source are investigated.

THE theory of the Cerenkov effect in crystals was developed as far back as 1940.[1] A number of interesting features of this radiation in optically anisotropic media (which have still not been investigated experimentally) have been pointed out in theoretical papers.[2,3] In recent years the Doppler effect in optically anisotropic media has also been analyzed theoretically.[4] The general features of radiation produced by various kinds of optical sources moving uniformly in refractory media have also been treated.[5] The critical velocity required by such sources for the production of various forms of radiation (Cerenkov radiation, complex Doppler effect, etc.) has been studied in detail only for an isotropic medium;[6] the case of an anisotropic medium has received only brief treatment.[5] In the present paper this problem is considered in greater detail.

In an optically isotropic medium, Cerenkov radiation is produced at a frequency ω when the velocity of the charge reaches the phase velocity of light $u(\omega) = c/n(\omega)$ for this frequency. This familiar condition comes directly from the relation between the frequency of the Cerenkov radiation and the direction of the wave vector:

$$(vn(\omega)/c)\cos\theta = 1. \qquad (1)$$

The minimum value of the angle θ between the wave vector and the velocity is zero; consequently the minimum velocity is given by $v = u(\omega)$.

The phase velocity $u(\omega)$ is just as important for the radiation of an arbitrary source moving in an optically isotropic medium. The radiation spectrum of a uniformly moving system characterized by a frequency $\omega_0 = \omega_0'\sqrt{1-\beta^2}$ is given by[6]

$$k(\omega)\cos\theta = (\omega \pm \omega_0)/v, \quad k(\omega) = \omega n(\omega)/c = \omega/u. \qquad (2)$$

Here $k(\omega)$ is the magnitude of the wave vector for frequency ω. In Eq. (2) the plus sign is taken for the so-called anomalous "superluminal" Doppler frequencies, if they are possible; the minus sign corresponds to the usual case of radiation in the medium. The Cerenkov radiation spectrum is obtained from Eq. (2) if we take $\omega_0 = 0$. If both sides of Eq. (2) are divided by ω, it becomes identical with Eq. (1) when $\omega_0 = 0$.

Equation (2) contains the phase velocity only, and is thus independent of the group velocity. This is natural, since the propagation of spectrally decomposed light, i.e., light at a given frequency ω, is determined completely by its phase velocity in an isotropic medium. Thus, the presence of a given frequency component in a spectrum (i.e., the conditions that must be satisfied for its production) depends on the phase velocity and not the group velocity. However, it is reasonable that the appearance of the radiation itself, not a given frequency component in the radiation, is to be associated with the group velocity.[5,6] It is reasonable to expect that the group velocity will play the same role for radiation in an optically anisotropic medium. However, the propagation of light in anisotropic media has a number of special features. For this reason the critical velocity of a radiator moving in an anisotropic medium requires special analysis.

An important factor in anisotropic media is that the phase velocity depends on the polarization of the wave as well as the direction of the vector $\mathbf{k}(\omega)$. We consider monochromatic waves of a given polarization, which propagate from some point. If we select waves for which the vector $\mathbf{k}(\omega)$ lies in a small solid angle (i.e., a narrow cone of normals to the wave surface), we find that the directional dependence of $k(\omega)$ is important as well as its magnitude. Because of this dependence, the direction of propagation of the waves, i.e., the direction of the ray, is not the same as the direction of the vector $\mathbf{k}(\omega)$. Hence the velocity $u'(\omega)$ is important; this is the velocity with which the phase of the wave is displaced in the direction of wave propagation, i.e., in the direction of the ray. Obviously, this velocity is given by

$$u'(\omega) = u(\omega) / \cos\alpha = c / n(\omega) \cos\alpha, \qquad (3)$$

where α is the angle between the ray and the normal to the wave (cf. Fig. 1).

If we lay off the velocity u' in the direction of wave propagation, i.e., in the direction of each ray, we obtain the so-called ray surface, which is the surface of constant phase for waves that emanate from a given center. In applying the Huygens principle to an anisotropic medium, the surface of the waves propagating from each point is actually obtained by plotting the ray surface to some appropriate scale. Thus, in problems involving the propagation of light in anisotropic media, the velocity u' plays the same role as the phase velocity in isotropic media. In particular, as is well known, the wave surface for Cerenkov radiation can also be obtained by means of the Huygens principle. It is not difficult to extend this procedure to an anisotropic medium and to verify in this case that the threshold for the appearance of a given frequency component ω is given by the relation $\mathbf{v} = \mathbf{u}'$.[5]

The analogy between u' and the phase velocity in an isotropic medium extends to the relation between u' and the group velocity w. If, from u' we form $k' = w/u'$, which is analogous to the wave vector, then

$$\partial k' / \partial\omega = 1 / w, \quad k' = \omega / u' = k \cos\alpha. \qquad (4)$$

Actually u' is the velocity of propagation of phase in the direction of the ray; the group velocity is in the same direction, so that $\partial k'/\partial\omega$ must have the same significance as $dk/d\omega$ in an isotropic medium. In this case, as in an isotropic medium, the notion of a group velocity is meaningful only in the frequency region in which the optical absorption is small. We shall assume that this condition is satisfied in the case at hand.

Because of its properties, u' might be called the second phase velocity. We shall call it the ray velocity.

The velocity u' also has an independent meaning, since we do not actually deal with a plane wave with a precisely defined direction, because true monochromatic waves do not exist. Just as a wave packet in some small frequency range propagates with a group velocity w, a packet of monochromatic waves in some angular range defines a ray velocity u'. In order to obtain plane waves, each of which is characterized by a vector $\mathbf{k}$, it is necessary to perform a double expansion, i.e., in frequency and in direction. Each of these partial waves is then characterized by the vector $\mathbf{k}$, (i.e., by the frequency) and by the phase-velocity vector. Hence, Eqs. (1) and (2), which contain the phase velocity, can be generalized to the case of optically anisotropic media if by k we understand the magnitude of this vector for a frequency ω and for a given direction of the normal to the plane of waves with a given polarization.

It should be noted, however, that the magnitude of k depends on the angle θ. Hence, in determining the value of v necessary for the production of a frequency component ω, we cannot proceed as in the case of an isotropic medium, i.e., we cannot simply take $\theta = 0$. The point is that when θ changes the quantity $n(\omega)$ also changes; consequently the maximum value of $n(\omega)\cos\theta$, which determines the minimum value of v in Eq. (1), is not necessarily obtained when $\theta = 0$. Actually, the maximum value of this quantity occurs when $\theta = \alpha$, where α is the angle between the direction of the ray and the vector $\mathbf{k}$. Comparing Eqs. (1) and (3) we see now that $v = u'$.

The critical-velocity problem for radiation in optically anisotropic media can be understood by an analysis of Figs. 1 and 2.

In Fig. 1 we show the surface for the vectors $\mathbf{k}(\omega)$ for an optically inhomogeneous medium,

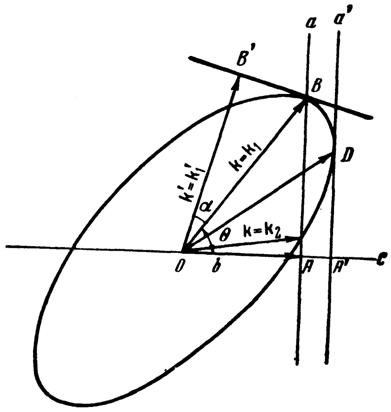

FIG. 1

for example for the extraordinary ray in a uniaxial crystal. If the $\mathbf{k}$ surface is given, it is easy to find the $\mathbf{k}'$ surface. To find the $\mathbf{k}'$ corresponding to a given $\mathbf{k} = k_1 = \overrightarrow{OB}$, we lay off the plane tangent to the surface at the point B. Then the normal OB' to this surface determines the magnitude and direction of the vector k'_1.*

It is easy to determine the directions of the vectors $\mathbf{k}(\omega)$ which satisfy Eq. (2) for a velocity v if we use the graphical construction proposed by Pafomov.[3,5] We assume that the velocity of the radiator with respect to the $\mathbf{k}$ surface is given by OC. In the direction OC we lay off a segment OA of length

$$b = (\omega \pm \omega_0)/v. \quad (5)$$

We assume here, as in Eq. (2), that the sign and magnitude of ω_0 are chosen in accordance with the conditions of the problem. For example, for ordinary Doppler frequencies we take ω_0 negative. If, in addition, $\omega < \omega_0$, the quantity b is negative and the segment OA must be laid off in the direction opposite to OC.

In order to find $\mathbf{k}$, we lay off the plane a, which passes through the point A and is perpendicular to the segment OA. This plane intersects the surface along some curve, which, as can be shown easily, represents the geometric locus of the ends of the $\mathbf{k}$ vectors which satisfy Eq. (2). For these vectors $k \cos\theta = b$, where θ is the angle between the given directions of $\mathbf{k}$ and $\mathbf{v}$ (cf. Fig. 1). Keeping in mind the value of b, we obtain Eq. (2) from Eq. (5) identically.

For fixed values of ω_0 and ω, b is inversely proportional to v. It is apparent that the plane a does not intersect the wave-vector surface for every value of b, that is, the frequency ω is not generated for every velocity. The critical velocity is obtained when plane a touches the surface. From Fig. 1 this condition corresponds to b = OA′, with plane a coinciding with a′. Under these conditions the $\mathbf{k}$ cone contracts about the direction OD. Since the plane a′ touches the surfaces at the point D, the normal to the plane, i.e., b = OA′, coincides with the direction of the ray conjugate to k = OD. In this case $\mathbf{b} = \mathbf{k}'$. Thus, from Eq. (5) we have

$$k'(\omega) = \frac{\omega}{u'^2}\mathbf{u}' = \frac{\omega \pm \omega_0}{v^2}\mathbf{v}. \quad (6)$$

In the particular case of Cerenkov radiation, i.e., for $\omega_0 = 0$, we immediately obtain from Eq. (5) the condition $\mathbf{v} = \mathbf{u}'$ for the critical velocity. Obviously Eq. (6) is a general condition which determines the velocity necessary for the appearance of a frequency component ω.[5] If b [right side of Eq. (6)] is positive, $\mathbf{u}'$ is in the same direction as $\mathbf{v}$. For negative values of b, these two vectors are in opposite directions. From Fig. (1), this condition corresponds to the case in which the plane a′ (tangent to the $\mathbf{k}$ surface) is to the left of the origin.

In the case shown in Fig. 1 the $\mathbf{k}$ surface is an ellipsoid. In order for a frequency component ω to appear in the spectrum the magnitude of b must be smaller than the corresponding radius vector of the ellipsoid, that is to say, v must be greater than some critical value. If the $\mathbf{k}$ surface were hyperbolic the reverse would hold. The quantity b would have to be greater than the corresponding radius vector and this means that the frequency component ω could be radiated only at a velocity smaller than the critical velocity.

In Fig. 1 we show the $\mathbf{k}$ surface for the extraordinary ray in a uniaxial crystal. In order to find the complete radiation pattern, a similar plot must also be made for the ordinary ray. In this case the $\mathbf{k}$ surface is a sphere, so that the cone of $\mathbf{k}$ vectors that satisfy Eq. (2) for the frequency ω is circular. As the velocity is reduced, this cone contracts about the direction of motion (or in the opposite direction for negative v). The critical velocity $v = u = u'$ for the ordinary rays differs from that for the extraordinary rays. The only exception is the case in which the radiator moves along the optical axis of the crystal.*

The radiation in a biaxial crystal exhibits a special feature. As is well known, the $\mathbf{k}$ surface cannot be represented by two independent surfaces in this case. The surface is more complicated in a biaxial crystal. However, a construction similar to that shown in Fig. 1 results in two $\mathbf{k}$ cones. The threshold velocities necessary for the production of each of these radiation cones are different, and when the velocity approaches the critical velocity the corresponding cone contracts about the direction of the $\mathbf{k}$ vector conjugate to the ray directed along the velocity (or opposite to it). Just as in

*It is easy to verify this construction. For small deviations of the vector $\mathbf{k}$ from $\mathbf{k}_1$, the end of the former slides along a plane which is tangent at point B. Hence the projection of vector $\mathbf{k}$ on OB′ is not sensitive to small changes in the wave vector. Whence it follows that the phases of a packet of plane waves characterized by vectors $\mathbf{k}$ which are almost in the direction of $\mathbf{k}_1$ are the same along the axis OB′. Thus interference effects cause addition of these waves in the direction OB′, i.e., $\mathbf{k}'_1 = \overrightarrow{OB'}$ is a ray which corresponds to the normal to the plane of the wave in the direction of $\mathbf{k}_1$.

*It should be recalled that Eq. (2) only gives possible directions for the vector $\mathbf{k}$ for radiation at a given frequency. For an actual radiation source the wave amplitudes for certain directions of $\mathbf{k}$ or even for the entire $\mathbf{k}$ cone can be zero. For example, in Cerenkov radiation, dipole radiation, or radiation from a linear multipole oriented in the direction of motion, there is no cone due to ordinary rays when the motion is parallel to the axis of the crystal, because waves of the appropriate polarization cannot be generated.

a uniaxial crystal, each direction of the **k** vector for waves of a given polarization is associated in this case with one direction of the ray, and vice versa.

The above considerations apply in all cases, except for cases in which the direction of the ray or the **k** vector is close to one of the optical axes of the crystal. In these cases, features analogous to internal and external refraction appear. These are as follows. It is well known that a ray which coincides with the optical axis of a biaxial crystal is to be associated with a **k**-vector cone rather than a single **k**. Hence, if a radiator moves along the axis of the crystal and its velocity approaches the critical velocity, the **k** cone does not contract about one direction, in contrast with the usual case. The threshold in this case is the **k**-vector cone associated with the ray directed along the axis (this cone is common for both forms of radiation in the crystal).

There is also an analog for internal refraction. In a biaxial crystal there is a direction for the **k** vector close to the direction of the axis which contains a whole cone of rays rather than one ray. Hence, if this direction satisfies Eq. (2), propagation takes place in the direction of the generatrices of the cone which is associated with this **k**. It is apparent that in this case the light emanates from the crystal in the form of a parallel beam of finite thickness. This feature, which is a consequence of internal refraction, has been pointed out by Ginzburg.[1]

Up to this point we have discussed radiation at a definite frequency ω. From what has been said it follows that the ray in whose direction a given frequency ω first appears (or disappears) as v changes is in the direction of **v** (or is in the opposite direction if b is negative). Actually a spectrum of frequencies is always radiated. For each of the frequencies there is a characteristic radiation cone, and the frequency spectrum associated with a given form of radiation (for example the spectrum of ordinary or "superluminal" Doppler frequencies, or any component of the complex Doppler effect) occupies some finite solid angle.

We now consider the conditions under which a given form of radiation is produced and the frequency is not given. We first consider the case of radiation in the direction of a ray that coincides with the direction of motion. In Fig. 2 we show the dependence of k′ on frequency ω for waves of a given polarization in a medium. The direction of the ray for which $k'(\omega) = k'_V(\omega)$ is taken to coincide with the direction of the velocity. In this

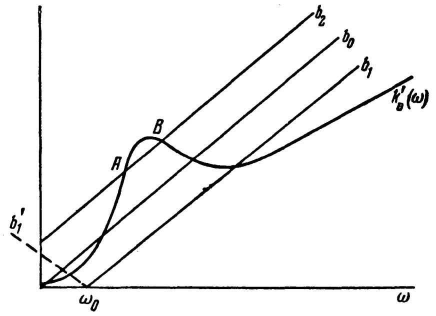

FIG. 2

same figure we draw lines corresponding to three possible values of b:

$$b_0 = \omega/v,\ b_1 = (\omega - \omega_0)/v,\ b_2 = (\omega + \omega_0)/v. \qquad (7)$$

It is apparent that the intersections of these lines with the $k'_V(\omega)$ curve yields the frequencies that satisfy Eq. (6), i.e., the frequency components radiated along the ray in the direction of the velocity; these are the threshold frequencies for a given velocity. The intersections with the line b_0 gives the Cerenkov radiation frequencies; intersections with b_1 give the ordinary Doppler frequencies, with b_2 the "superluminal" Doppler frequencies.

Figure 2, as can easily be shown, is completely analogous to Fig. 1 of reference 6, which shows the features of radiation in an optically isotropic medium. The only difference is the fact that there the angle θ is arbitrary while here it is fixed in such a way as to give the direction of the ray (i.e., $\theta = \alpha$). The conclusions that can be obtained from an analysis of the curves will obviously be the same in both cases. They are the following. The slope of $k'_V(\omega)$ is, in accordance with Eq. (4), $dk'_V/d\omega = 1/w$. Here w is the group velocity for frequency ω for a ray in the direction of the velocity. The slope of the line b is 1/v. When v changes the slope of the line changes and the point at which it intersects the curve is displaced, that is to say, the threshold frequency is changed. The intersection vanishes or appears for those values of v for which the line b is tangent to the $k'_V(\omega)$ curve. In order for the line b to be tangent it is necessary that at the point of tangency the slope of $k'_V(\omega)$, i.e., $dk'_V(\omega)/d\omega$, be equal to 1/v. Thus, the production of radiation or new components requires that

$$v = w(\omega_{lim}), \qquad (8)$$

where ω_{lim} is the frequency which first appears in the radiation spectrum [this frequency must satisfy Eq. (6)].

The quantity b_1 can be negative if $\omega < \omega_0$. In this case, in Eq. (6) we must substitute $k'(\omega)$ for a ray directed in the direction opposite to the velocity and make $k'(\omega)$ negative. Changing the signs on both sides of Eq. (6), we then obtain

$$k'_{-v}(\omega) = (\omega_0 - \omega)/v. \tag{6a}$$

Here $\mathbf{k}'_{-\mathrm{v}}(\omega)$ is the vector $\mathbf{k}'$ for the ray directed opposite to the velocity vector. If a reversal of direction of the ray does not change the magnitude of $\mathbf{k}'$, as is the case in a usual medium, the solution of Eq. (6a) can again be found by means of Fig. 2. The sought Doppler frequency is obtained as the intersection of the line $b'_1 = (\omega_0 - \omega)/v$ (dashed line in Fig. 2) with the $k'_V(\omega)$ curve. As is apparent from the figure, an intersection of this kind always exists, corresponding to the obvious fact that in the ordinary Doppler spectrum there is a frequency component which can be radiated in the direction opposite to the direction of motion. In principle, more than one intersection is possible (complex Doppler effect). Just as for positive b, the threshold for the complex Doppler effect requires (beside intersection) that the line b_1 be tangent to the $k_{-V}(\omega)$ curve. Repeating the analysis given above we find that in this case the group velocity must be negative and equal to v. The negative sign means that the group velocity is opposite to the vector $\mathbf{k}'_{-\mathrm{V}}(\omega)$, i.e., the velocity. Thus, the condition for the appearance of a new radiation component is again given by Eq. (8).

All of these results are completely analogous to those obtained earlier for an isotropic medium (cf. references 5 or 6). If we consider the possible intersections of the lines b with the $k'_V(\omega)$ curve it is apparent (as in an isotropic medium) that the presence of a frequency ω' for which $w(\omega') < v$ means that there must also be a frequency ω'' for which $w(\omega'') > v$, that is to say, the composition of the radiation is necessarily complex. Let us assume that at the point of intersection of the line b with the $k'_V(\omega)$ curve the curve rises more sharply than the line, i.e., $dk'_V/d\omega = 1/w > 1/v$ (cf. for example point A on the line b_2). This means that at the point of intersection the line moves from the region above the $k'_V(\omega)$ curve into the region of the plane under the $k'_V(\omega)$ curve. Then, at higher frequencies there must be at least one intersection for which $dk'_V/d\omega = 1/w < 1/v$ (point B in Fig. 2). At this intersection, the line moves from the region under the curve to the region above the curve. That this is true follows from the fact that for sufficiently high ω the tangent to the $k'_V(\omega)$ curve approaches $1/c$ (in the limit $n = 1$ and $\alpha = 0$) and, since $1/c < 1/v$, all lines b for high ω must be above the $k'_V(\omega)$ curve.

We may note that the meaning of the provisional term "complex radiation" for an anisotropic medium is somewhat different than the meaning of this term for an optically isotropic medium. In an isotropic medium complex radiation means that there are several frequency components for a set of values of v, ω_0, and θ. In the anisotropic medium there are two radiation cones for the two polarization directions. In addition, there are two other possibilities: waves at different frequencies may have the same directions for $\mathbf{k}$, i.e., coincident wave normals, or waves at different frequencies may have the same ray directions. As is apparent from the above, here we have considered the case in which the several frequencies for a given form of radiation are characterized by the same ray direction.

In this paper we have considered only the particular case in which the direction of the ray coincides with the direction of the velocity (or is opposite to it). Complex radiation effects are obviously also possible for other ray directions.

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