COLLECTIVE EXCITATIONS OF NON-AXIAL EVEN-EVEN NUCLEI

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The collectively excited states of non-axial even-even nuclei are investigated with account of the interaction between rotational and vibrational states.

DAVYDOV and Filippov[1] have calculated the collective energy levels of axially symmetric even-even nuclei with account of the interaction between rotational and vibrational states. The same authors[2] also investigated the rotational energy levels and the probabilities of transitions between them for the case of non-axial nuclei. The coupling between the vibrations and the rotation was not taken into account. In the present paper we study the collective excitations of the vibration-rotation type in non-axial even-even nuclei.

1. ENERGY LEVELS

We shall assume that the energy of the vibrations in the parameter of axial asymmetry, γ, is much larger than the energies of the collective rotation and of the vibrations in β. γ can then be regarded as constant. According to the collective model of Bohr, the energy of the nucleus is made up of the following terms:

$$E_0 = \tfrac{1}{2} B\dot{\beta}^2 + \tfrac{1}{2} C\beta^2 + A\beta + \sum_{\chi} M_{\chi}^2/2I_{\chi}. \tag{1}$$

The first two terms represent the kinetic and potential energy of the β oscillations of the nucleus, respectively; $A\beta$ is the interaction of the external nucleons with the nuclear core, and is in first approximation proportional to β; the last term represents the energy of rotation of the nucleus, where M_χ is the projection of the classical angular momentum on the χ axis, and I_χ is the moment of inertia, which is given by the expression

$$I_{\chi} = 4B\beta^2 \sin^2(\gamma - 2\pi\chi/3).$$

In going from Eq. (1) to the corresponding quantum-mechanical equation, we have

$$\hat{H}\Psi = E_0\Psi, \tag{2}$$

where

$$\hat{H} = -\frac{\hbar^2}{2B}\frac{1}{\beta^3}\frac{\partial}{\partial\beta}\beta^3\frac{\partial}{\partial\beta} + \frac{C\beta^2}{2} + A\beta + \hat{L},$$
$$\hat{L} = \sum_{\chi}\frac{\hbar^2}{2I_{\chi}}\Big(\hat{J}_{\chi} - \sum_n \hat{j}_{\chi}(n)\Big)^2, \tag{3}$$

$\hat{J}_\chi$ is the projection of the total angular momentum of the nucleus on the χ axis, and $j_\chi(n)$ is the projection of the angular momentum of the n-th external nucleon.

The volume element in the space (β, θ_i) is written as

$$d\tau = G\beta^3 \sin\theta \, d\beta \, d^3\theta \tag{4}$$

(G is a constant).

In the adiabatic approximation with respect to the one-nucleon states we can average the Hamiltonian operator (3) over the states of the external nucleons. If I is the sum of the spins of the external nucleons and Ω is its projection on the axis fixed in the nucleus, we can use the conditions of symmetry to obtain the wave function of the external nucleons in the form

$$\Psi_n = \sum_{I,\Omega} a_{I\Omega}\left(\Phi_{I\Omega} + (-1)^I \Phi_{I,-\Omega}\right), \tag{5}$$

where Ω takes on only even values. The average value of the operator of the rotational energy is equal to

$$\langle \Psi_n | \hat{L} | \Psi_n \rangle = \hbar^2 D/6B\beta^2 + \sum_{\chi} \hbar^2 \hat{J}_{\chi}^2/2I_{\chi},$$
$$\hbar^2 D/6B\beta^2 = \sum (\hbar^2/2I_{\chi}) \langle \Psi_n | \Big(\sum_n \hat{j}_{\chi}(n)\Big)^2 | \Psi_n \rangle,$$

so that, using Eq. (5),

$$\langle \Psi_n \Big| \sum_n \hat{j}_{\chi}(n) \Big| \Psi_n \rangle = 0.$$

We shall regard $\hbar^2 D/6B\beta^2$ as being included in the effective potential energy. Then the operator of the energy of rotation is $\sum_{\chi} \hbar^2 J_{\chi}^2/2I_{\chi}$. We seek the solution of Eq. (2) in the form of a product of the wave functions for the β oscillations (Ψ_β) and for the collective rotation (Ψ_{rot}). For fixed β we have

$$\sum_\chi \frac{\hbar^2 \hat{J}_\chi^2}{2I_\chi}\Psi_{rot} = \frac{\hbar^2}{4B\beta^2}\tilde{\varepsilon}_{J\lambda}\Psi_{rot}; \qquad (6)$$

J is the total angular momentum, and λ is the number of the solution for a given J. The quantities $\tilde{\varepsilon}_{J\lambda}$ were obtained by Davydov and Filippov[2] using the symmetry conditions for Ψ_{rot}. With the help of their data we write Eq. (2) in the form

$$\left\{-\frac{\hbar^2}{2B}\frac{1}{\beta^3}\frac{\partial}{\partial\beta}\beta^3\frac{\partial}{\partial\beta} + V'(\beta) + \frac{\hbar^2}{4B\beta^2}\tilde{\varepsilon}_{J\lambda}\right\}\Psi_\beta = E_0\Psi_\beta,$$

$$V'(\beta) = C\beta^2/2 + A\beta + \hbar^2 D/6B\beta^2. \qquad (7)$$

Substituting the value $\Psi_\beta = U_{J\lambda}(\beta)/\beta^{3/2}$ in Eq. (7), we obtain the equation

$$\left\{-\frac{\hbar^2}{2B}\frac{d^2}{d\beta^2} + V(\beta) + \frac{\hbar^2}{4B\beta^2}\tilde{\varepsilon}_{J\lambda}\right\}U_{J\lambda} = E_0 U_{J\lambda},$$

where the quantity

$$V(\beta) = V'(\beta) + 3\hbar^2/8B\beta^2$$

reaches a minimum in the point

$$\beta_0 = -A/C + \hbar^2 D/3BC\beta_0^3 + 3\hbar^2/4BC\beta_0^3.$$

Expanding $V(\beta)$ in a series in terms of β_0, we find

$$\left\{-\frac{\hbar^2}{2B}\frac{d^2}{d\beta^2} + \frac{C_0}{2}(\beta-\beta_0)^2 + \frac{\hbar^2}{4B\beta^2}\tilde{\varepsilon}_{J\lambda} - \varepsilon\right\}U_{J\lambda} = 0,$$

$$\varepsilon = E_0 - V(\beta_0), \quad C_0 = C + \hbar^2 D/B\beta_0^4 + 9\hbar^2/4B\beta_0^4. \qquad (8)$$

Equation (8) agrees with that obtained by Davydov and Filippov,[1] except that $J(J+1)$ is replaced by $\frac{3}{2}\tilde{\varepsilon}_{J\lambda}$. [For $\gamma \to 0$ we have for the lowest level for a given J: $\frac{3}{2}\tilde{\varepsilon}_{J\lambda} \to J(J+1)$].

The energy of rotation-vibration of the nucleus is determined by the equation

$$\varepsilon_\nu(J\lambda)/\hbar\omega_0 = (\nu + \tfrac{1}{2})\sqrt{1 + \tfrac{3}{2}\tilde{\varepsilon}_{J\lambda}/\delta^4\xi^4}$$

$$+ \tfrac{1}{2}\delta^2(\xi-1)^2 + \tilde{\varepsilon}_{J\lambda}/4\delta^2\xi^2, \qquad (9)$$

where

$$\delta = \beta_0(BC/\hbar^2)^{1/4}, \quad \xi^3(\xi-1) = \tilde{\varepsilon}_{J\lambda}/2\delta^4, \quad \omega_0 = \sqrt{C_0/B}.$$

The quantity ν is found from the equation

$$H_\nu(-\delta_1\xi) = 0 \text{ for } \delta_1 = \delta(1 + \tfrac{3}{2}\tilde{\varepsilon}_{J\lambda}/\delta^4\xi^4)^{1/4}, \qquad (10)$$

where

$$H_\nu(\zeta) = \frac{1}{2\Gamma(-\nu)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma\left(\frac{k-\nu}{2}\right)(2\zeta)^k$$

is a Hermite function of the first kind.

The unnormalized wave function Ψ_β has the form

$$\Psi_\beta = \beta^{-3/2}e^{-\zeta^2/2}H_\nu(\zeta), \quad \zeta = \delta_1(\beta/\beta_0 - \xi). \qquad (11)$$

We shall be interested only in the excitation energy $E_{J\lambda} = \epsilon_{J\lambda} - \epsilon_{J=0}$. For large values of δ (usually already for $\delta > 3$) the correction to the energy due to the coupling of the rotation with the β oscillations can be obtained with the help of an expansion in terms of the small parameter $1/\delta^4$. Then*

$$E_{J\lambda} = (\hbar^2/4B\beta_0^2)\tilde{\varepsilon}_{J\lambda} - F(\tilde{\varepsilon}_{J\lambda})^2 + M\tilde{\varepsilon}_{J\lambda},$$

$$F = \text{const}, \qquad M = \text{const}.$$

The quantity ν in Eq. (10) runs through an infinite sequence of discrete values. Each branch with J = 0 has its set of levels with spins 2, 3, 4, 6, etc. If $\delta > 2.5$, the lowest levels of the two lowest branches split up and we obtain a vibrational-rotational band (cf. reference 1).

We studied the quantities $R_\lambda(J) = E_{J\lambda}/E_1(2^+)$ as functions of $R_2(2) = E_2(2^+)/E_1(2^+)$. We considered the two lowest levels 4^+, the first level 6^+, and the level 3^+ corresponding to the first band, and also the level 0^+ of the second band. The results of the calculations for $R_1(6)$ and $R_1(4)$ are shown in Figs. 1 and 2. The points with δ = 0, 1, 2, 3, 4, and ∞ and with γ = 10, 15, 22.5, and 30° are joined by curves. The curve with $\delta = \infty$ corresponds to the absence of the coupling between the rotation and the β oscillations [the excitation of the β oscillations is infinitely high as compared with $E_1(2^+)$]. It should be noted that $R(3)$ varies little for a given $R_2(2)$, i.e., the equality $E_1(2^+) + E_2(2^+) \approx E(3^+)$ is approximately conserved.

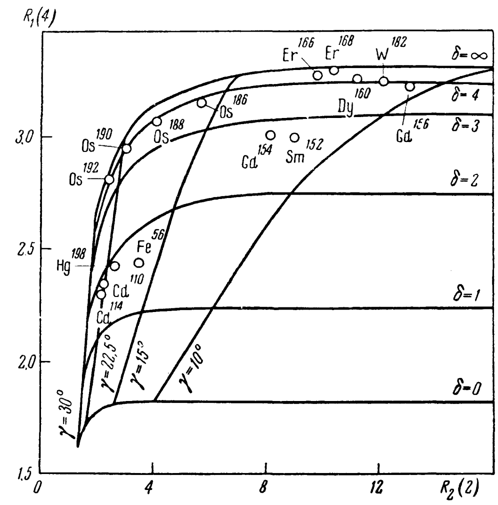

FIG. 1

*An empirical formula of this form (with M = 0) was proposed by Malman and Kerman (preprint) for the comparison of the theory of Davydov and Filippov[2] and of Davydov and Rostovskiĭ[3] with the experimental data.

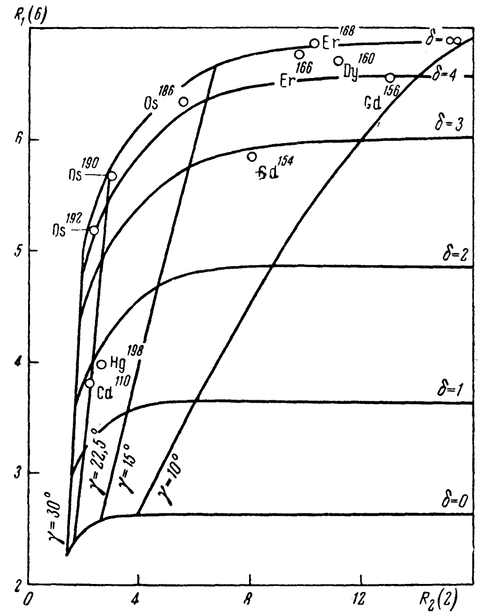

FIG. 2

The curves with small δ lie below the curves with larger δ for all levels; therefore, $R_\lambda(J)$ decreases if the coupling between the oscillations and the rotation is included.

2. PROBABILITIES OF QUADRUPOLE TRANSITIONS

The expression for the reduced probability for a quadrupole transition between the states $J\lambda$ and $J'\lambda'$ has the form

$$B(E2; J\lambda \to J'\lambda') = \frac{5}{16\pi(2J+1)} \sum_{M,m,\mu} |(\Psi_{\text{rot}}^{J'\lambda' m} \Psi_{\beta}^{J'\lambda'} | \hat{Q}_{2\mu} | \Psi_{\text{rot}}^{J\lambda M} \Psi_{\beta}^{J\lambda})|^2,$$

$$\hat{Q}_{2\mu} = e Q_0 \frac{\beta}{\beta_0} \left(D^2_{\mu 0} \cos\gamma + \frac{1}{\sqrt{2}} (D^2_{\mu 2} + D^2_{\mu -2}) \sin\gamma\right),$$

$$Q_0 = \frac{3}{\sqrt{5\pi}} Z R^2 \beta_0. \qquad (12)$$

We rewrite Eq. (12) in the form

$$B(E2; J\lambda \to J'\lambda') = \widetilde{B}(E2; J\lambda \to J'\lambda')\, \beta_0^{-2}\, |\langle J'\lambda' | \beta | J\lambda \rangle|^2, \qquad (13)$$

where $\widetilde{B}$ is a quantity which was computed in reference 2. The correction factor represents the matrix element of β/β_0 with respect to the functions Ψ_β. Using (11) and (4), we have

$$\langle i | \beta | j \rangle = \int_0^\infty U_i(\beta) U_j(\beta) \beta d\beta \Big/ \left\{\int_0^\infty U_i^2(\beta) d\beta\right\}^{1/2} \left\{\int_0^\infty U_j^2(\beta) d\beta\right\}^{1/2}. \qquad (14)$$

Here

$$U_i(\beta) = \exp(-\zeta_i^2/2) H_{\nu_i}(\zeta_i), \quad \zeta_i = (\delta_1)_i (\beta/\beta_0 - \xi_i).$$

For $\delta \to \infty$ we have $\nu_i \to 0$. In practice $\nu_i = 0$ for $\delta \geq 2$. (ν has the largest value in the case $J = 0$. If $J = 0$, $\delta = 2$, then $\nu = 0.02$.) For $\delta \geq 2$ the quantity H_{ν_i} can be replaced by $H_0 = 1$. Then expression (14) has a very simple form, which leads to the ratio ($\delta \geq 2$)

$$\frac{B(E2; 22 \to 21)}{B(E2; 22 \to 0)} = \mu \frac{\widetilde{B}(E2; 22 \to 21)}{\widetilde{B}(E2; 22 \to 0)};$$

$$\mu = \frac{\langle 21 | \beta | 22 \rangle^2}{\langle 0 | \beta | 22 \rangle^2} = \frac{(\delta_1)_1}{\delta} \exp\left\{2\left(\frac{b_{21}^2}{4a_{21}} - d_{21}\right) - 2\left(\frac{b_{20}^2}{4a_{20}} - d_{20}\right)\right\}\left(\frac{b_{21}}{b_{20}}\right)^2 \left(\frac{a_{20}}{a_{21}}\right)^3, \qquad (15)$$

$$a_{ij} = \frac{1}{2}[(\delta_1)_i^2 + (\delta_1)_j^2], \quad b_{ij} = (\delta_1)_i^2 \xi_i + (\delta_1)_j^2 \xi_j,$$

$$d_{ij} = \frac{1}{2}[(\delta_1)_i^2 \xi_i^2 + (\delta_1)_j^2 \xi_j^2]. \qquad (16)$$

The subscripts 0, 1, and 2 refer to the ground state, the lowest excited state with $J = 2$, and the second excited state with $J = 2$, respectively.

The most abundant group of nuclei experimentally has $\delta \sim 4$ and $10° \leq \gamma \leq 30°$. In this case $\mu \approx 1.015$ (it varies within the limits 1.013 and 1.017 for $\delta = 4$). For the second group $\delta \sim 2$ and $15° \leq \gamma \leq 30°$. Here $\mu \approx 1.2$ (it varies within the limits 1.20 and 1.23 for $\delta = 2$). If $\gamma \to 0$ for fixed δ, $\mu \to 0$; but this refers already to very small γ. If $\delta \to \infty$, then $\mu \to 1$, as should be expected ($\delta \to \infty$ corresponds to the absence of coupling between the β oscillations and the rotation). The correction due to the factor μ is thus important only for nuclei with $\delta \sim 2$. But here (as in the case of nuclei with $\delta \sim 4$) the change of γ on account of the coupling between the β oscillations and the rotation plays an incomparably greater role.

3. DISCUSSION OF THE RESULTS

The comparison of the theoretical results with experiment will be carried out in the following way. We determine δ and γ from the three levels $E_1(2^+)$, $E_2(2^+)$, and $E_1(4^+)$ with the help of Fig. 1, which shows the dependence of $R_1(4)$ on $R_2(2)$. It is here convenient to use curves with fixed values of δ and curves with fixed values of γ. With the help of the analogous curves computed for the dependence of $R_1(6)$ (see Fig. 2), $R_2(4)$, and $R_2(0)$ on $R_2(2)$ we can find $R_1(6)$, $R_2(4)$, and $R_2(0)$ from the known values of δ and γ. The results of the comparison with experiment are given in Table I (see also Figs. 1 and 2). The character of the interaction between the β oscillations and the rotation is apparently well accounted for by the calculations.

TABLE I. Energies of the excited states of the vibrational-rotational type*

Nucleus	E_1 (2+), kev	R_2 (2)	R (3)	R_1(4)	R_2 (4)	R_1 (6)	R_2 (0)	Reference
				2.47	4.85			
Fe^{56}	845	3.49	4.54	2.47	4.7	4.7	2.9	[4]
				3.09	6.2	5.94		
				2.35		3.80		
Cd^{110}	656	2.10	3.31	2.35	3.8	3.8	2.8	[5,6]
				2.78	5.5	5.23		
				2.30			2.35	
Cd^{114}	556	2.18		2.30	3.8	3.7	2.7	[7]
				2.75	5.5	5.14		
				3.01	11.68		5.61	
Sm^{152}	121.8	8.92	10.14	3.01	10.3	5.7	6.0	[7,8]
				3.31	11.3	6.80		
				3.02		5.84	5.52	
Gd^{154}	123.02	8.10	9.17	3.02	9.6	5.7	6.2	[7,8]
				3.30	10.5	6.75		
				3.24	15.34	6.56	~8.09	
Gd^{156}	89	13.01	14.00	3.24	14.9	6.5	13.6	[7,8,9]
				3.32	15.4	6.88		
				3.27	13.35	6.70		
Dy^{160}	86.6	11.16	12.11	3.27	13.2	6.6	>14	[7,9]
				3.32	13.6	6.86		
				3.29	11.87	6.76	18.12	
Er^{166}	80.7	9.76	10.67	3.29	12.0	6.7	>14	[7,10]
				3.31	12.2	6.82		
				3.31	12.47	6.86		
Er^{168}	79.9	10.29	11.22	3.31	12.7	6.8	>14	[9,10]
				3.32	12.8	6.84		
				3.26	14.68			
W^{182}	100.9	12.11	13.20	3.26	14.0	6.6	>14	[7,8]
				3.32	14.5	6.87		
				3.16	7.73	6.33		
Os^{186}	137.2	5.60	6.63	3.16	7.7	6.2	11.4	[8,10]
				3.23	8.1	6.49		
				3.08	6.17		7.01	
Os^{188}	155.0	4.09	5.10	3.08	6.3	5.8	12.1	[7,8,10]
				3.15	6.7	6.14		
				2.95	5.14	5.63		
Os^{190}	186	2.99	4.05	2.95	5.6	5.5	11.2	[8]
				3.01	5.8	5.75		
				2.82		5.17		
Os^{192}	206	2.37	3.35	2.82	5.4	5.2	>14	[7,9]
				2.84	5.5	5.34		
				2.43		3.97	2.63	
Hg^{198}	411.8	2.65		2.43	4.1	4.0	2.9	[7,9]
				2.92	5.6	5.53		
				3.30		6.86	21.15	
Pu^{238}	44.2	23.30	24.34	3.30		6.84	>14	[7]
				3.33		7.00		

*For each element we list in the first row the experimental values, in the second row the theoretical values, and in the last row the values obtained when only the rotational excitations are taken into account.

In Table II we give the ratios of the reduced probabilities for the transitions $22 \rightarrow 21$ and $22 \rightarrow 0$. The effect of μ is not accounted for because of its smallness. Indeed, the change of the value of γ on account of the coupling between the vibrations and the rotation plays a much larger role. But for nuclei with $\delta \sim 4$ even this effect does not change greatly the value found by Davydov and Filippov in reference 2 (γ becomes somewhat smaller). For nuclei with $\delta \sim 2$ the value of γ becomes appreciably smaller. The experimental data at our disposal are too inaccurate and very sparse in the region $\delta \sim 2$. They do not permit us to check the results of the theory with regard to the transition probabilities.

In conclusion I regard it my obligation to thank

TABLE II. Ratio of the reduced probabilities for quadrupole transitions from the second level 2 to the first level 2^+ and from the second level 2^+ to the ground state

Nucleus	γ, degrees	γ^*, degrees	$B(E2;\ 22 \rightarrow 21)/(B(E2;\ 22 \rightarrow 0)$ Theory	Rotation only	Experiment
Os^{186}	15.9	16.5	3.05	3.30	2.5; 5 [11]
Os^{188}	18.5	19.2	4.36	4.84	2.7 [3] 3.3 [11]
Os^{190}	21.9	22.2	7.8	8.3	9.7 [3] 5.6 [11]
Os^{192}	24.5	25	17.5	20.6	9.1 [3]
W^{182}	11.0	11.45	1.81	1.88	1.59 [2]
Gd^{154}	11.85	13.9	1.94	2.38	1.94 [3] 2.2 [11]
Gd^{156}	10.5	11.0	1.75	1.81	1.7 [11]
Sm^{152}	11.42	13.25	1.87	2.42	1.7 [2] 2.4 [11]
Dy^{160}	11.45	11.9	1.88	1.94	2.38 [2]
Cd^{114}	23.3	26.7	11.3	52	77.9 [2]
Hg^{198}	21.0	23.6	6.5	12.7	25 [2] 67 [9]

*Rotation only.

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