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ON THE AMOUNT OF ACCELERATED PARTICLES IN AN IONIZED GAS UNDER VARIOUS ACCELERATING MECHANISMS

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The kinetic equation is solved for a system of particles interacting according to Coulomb-law conditions where the presence of the accelerating mechanism makes the distribution function non-stationary in the high-velocity range, i.e., when the accelerating mechanism leads to continuous acceleration of particles whose energy is greater than the injection energy. The value of the particle flux is determined.

1. Great attention is being paid of late, in connection with investigations of the origin of cosmic rays and corpuscular radiation from the sun, to the analysis of various mechanisms of acceleration in an ionized gas. Under consideration, for example, is Fermi's statistical mechanism of acceleration, by which the ion energy increases through collisions with clouds of charged particles. In this case, if the free path L between two collisions of an ion with the clouds is constant (independent of the ion energy ϵ), and if the ion acquires during each impact an average energy $(2/3)\, Mv_c^2$ (where M is the ion mass and v_c the average cloud velocity), then the energy transferred to the ion by the accelerating mechanism over a time Δt is

$$\Delta\varepsilon_+ = \frac{2}{3} Mv_c^2 \frac{\sqrt{2\varepsilon/M}}{L} \Delta t.$$

Within the same time, the particle loses, by interaction with other ions of the plasma, an energy $\Delta\epsilon_- = (\epsilon - 3kT/2)\,\nu(\epsilon)\,\Delta t$, where $\nu(\epsilon)$ is the collision frequency. As is well known, the collision frequency of the particles interaction in Coulomb-law diminishes rapidly with increasing energy, $\nu(\epsilon) = \nu_0 (kT/\epsilon)^{3/2}$, where

$$\nu_0 = \nu(kT) = \frac{4\pi N e^4}{(kT)^{3/2} M^{1/2}} \ln\left(\frac{kTD}{e^2}\right) \qquad (1)$$

is the collision frequency of a particle of average (thermal) energy (e is the ion charge, N the ion density, and D the Debye radius). Thus, the change in ion energy, due to the Fermi statistical acceleration mechanism, is given by

$$\frac{d\varepsilon}{dt} = \frac{2}{3} Mv_c^2 \frac{\sqrt{2\varepsilon/M}}{L} - \nu_0\left(\varepsilon - \frac{3}{2}kT\right)\left(\frac{kT}{\varepsilon}\right)^{3/2}, \qquad (2)$$

from which we see directly that only when

$$\varepsilon < \varepsilon_{in} = 3\nu_0 L\,(kT)^{3/2}/2\sqrt{2} v_c^2 M^{1/2}$$

will the ion energy tend to a stationary value. When $\epsilon > \epsilon_{in}$ it increases continuously with time, and in collisions with other ions the particle loses only a small part of the energy it acquires from the clouds. ϵ_{in} is usually called the injection energy. It is clear that when the injection energy is close to the average thermal ion energy ($\epsilon_{in} \sim kT$), all particles are immediately accelerated. If $\epsilon_{in} \gg kT$, on the other hand, only a small part of the particles become accelerated, those with sufficiently large energies $\epsilon > \epsilon_{in} \gg kT$.

When $\epsilon_{in} \gg kT$, the acceleration mechanism is logically called "weak." We shall consider here only weak accelerating mechanisms. Since a weak acceleration mechanism transfers to the ions, over the mean free time $1/\nu_0$, an energy which is small compared with the thermal energy kT, the action of a weak acceleration mechanism cannot cause the distribution function to deviate considerably from equilibrium (Maxwellian). In the range of high energies, however, the mean free time is large, and the effect of a weak acceleration mechanism can be substantial. Furthermore, when $\epsilon > \epsilon_{in}$ the particle energy increases with time, so that the particle energy distribution in this region is essentially not in equilibrium.

Collisions cause the energy of certain particles to increase from the main equilibrium value to the injection value. The particles then leave (escape) the main region and are accelerated. The order of magnitude of the number of particles escaping per unit time Δt is, naturally,

$$\Delta N \sim \nu \Delta t N(\varepsilon_{in}), \qquad (3)$$

where $N(\epsilon_{in})$ is the number of particles with energies ranging from $\epsilon_{in} - kT$ to ϵ_{in}. It appears natural at first glance to assume, in estimating the value of $N(\epsilon_{in})$, that the distribution function is Maxwellian up to $\epsilon = \epsilon_{in}$. We would obtain then

$$\Delta N \sim \nu \Delta t N_0 \exp\{-\varepsilon_{in}/kT\}. \quad (3a)$$

We shall see below that the acceleration mechanism changes the character of the velocity distribution in the region $\epsilon \sim \epsilon_{in}$ so much that expressions of type (3) are utterly inapplicable for estimates of the number of escaping particles.

At the same time, the number of escaping particles, and consequently the number of accelerated particles, is one of the basic characteristics of the acceleration mechanism. Therefore the question of the flux of escaping particles and its dependence on the parameters that characterize the acceleration mechanism, the plasma, and the accelerating particles themselves is of considerable interest.

The present paper is devoted to a solution of the corresponding problem.* In Sec. 2 we develop a method for finding a weakly-nonstationary solution of the kinetic equation. A general expression is obtained for the flux of escaping particles. This solution is analyzed in Sec. 3 for ions accelerated in a plasma by the statistical Fermi acceleration mechanism.

2. We consider first, for the sake of simplicity, a system of identical particles, interacting in accordance with the Coulomb law and subject to a weak statistical acceleration mechanism. The equation for the velocity distribution function of these particles has the form

$$\frac{\partial f}{\partial \tau} - \frac{1}{u^2}\frac{\partial}{\partial u}\left\{\left(\frac{1}{u} + u^2\alpha(u)\right)\frac{\partial f}{\partial u} + f\right\} = 0. \quad (4)$$

Allowance is made here for the fact that the statistical acceleration mechanism does not have a preferred direction in velocity space; the distribution function can therefore depend only on the modulus of the velocity v and on the time $t\,[f = f(v, t)]$. In addition, we use in (4) the dimensionless variables

$$u = v/\sqrt{kT/M}, \qquad \tau = t\nu_0, \quad (5)$$

where ν_0 is the collision frequency, determined by Eq. (1), while T is the system temperature (T is considered henceforth constant in time; this is meaningful only in the case of a weak acceleration mechanism).* Finally, the function $\alpha(u)$ describes the effect of the acceleration mechanism on the particles. Thus, if the mean free path L of the particle between two collisions with the clouds is independent of its velocity, then

$$\alpha(u) = \alpha_0 u, \qquad \alpha_0 = {}^2/_3\, v_c^2\, /\nu_0 L \sqrt{kT/M}. \quad (6)$$

The parameter α_0 is very simply connected with the injection energy ϵ_{in}, $\alpha_0 = kT/\sqrt{2}\,\epsilon_{in}$; in the case of a weak acceleration mechanism, α_0 is always much less than unity.

STATIONARY SOLUTION

Solving Eq. (4) under stationary conditions $(\partial f/\partial \tau = 0)$, we have

$$(1/u + u^2\alpha(u))\,\partial f/\partial u + f = -S_0, \quad (7)$$

where S_0 is the total flux of the particles through the surface $u = \text{const}$ in velocity space.† Hence

$$f = C_0 \exp\left\{-\int_0^u \frac{u\,du}{1 + u^3\alpha(u)}\right\} - S_0, \quad (8)$$

where the constant C_0 is determined by the normalization condition

$$\int_0^\infty u^2 f\,du = N_0. \quad (8a)$$

The flux S_0 can differ from zero in the entire velocity space only if a source is located at the origin.

In the absence of such a source we have for $\alpha(u) = \alpha_0 u$ [see Eq. (6)]

$$f = C_0 \exp\left\{-\int_0^u \frac{u\,du}{1 + \alpha_0 u^4}\right\}. \quad (9)$$

The distribution function (9) tends to a constant value as $u \to \infty$: $f(\infty) = C_0 \exp(-\pi/4\sqrt{\alpha_0})$. It cannot satisfy, naturally, the normalization condition (8a). This means that there exists no stationary distribution in this case, as should be

*An analysis of the question of the number of accelerating ions in a plasma in a case of the Fermi acceleration mechanism is treated by Parker.[1] He did not obtain, however, correct expressions for the flux of the escaping particles. We note also that Parker disregarded the interaction with electrons, which fundamentally affects the flux of escaping ions (see Sec. 3 of the present paper).

*We note that (4) holds, strictly speaking, only for particles much faster than thermal ($u^2 \gg 1$). In the vicinity of $u \sim 1$, the terms that describe collisions between particles should be written down in somewhat more complicated form (see reference 2). It is easy to verify, however, that this is of no importance to us, for in the case of an acceleration mechanism only the region of high velocities is of importance.

†Equation (4) can be written in the form $\partial f/\partial \tau + \operatorname{div} S = 0$, from which it is clear that $S = \int S\,d\boldsymbol{\sigma}$ (integration over the region $u = \text{const}$) is actually the particle flux in velocity space.

(see Sec. 1). We note that if $\alpha(u) = \alpha_0 u^\lambda$, then no normalizable stationary distribution exists for $\lambda < -1$, and the particles do not escape in this case. A similar conclusion is reached by analysis of Eq. (2), where the acceleration of particles for $\Delta\epsilon_+ \sim \epsilon^{\lambda/2}$ is possible only if $\lambda > -1$.

We note in conclusion that in the general case when the equation for the function f has the form

$$\frac{\partial f}{\partial \tau} - \frac{1}{u^2}\frac{\partial}{\partial u}\left(A(u)\frac{\partial f}{\partial u} + B(u)f\right) = 0, \tag{10}$$

the stationary solution (with flux S_0) is given by

$$f = C_0 \exp\left\{-\int_0^u \frac{B(u)}{A(u)}du\right\} - S_0 \exp\left\{-\int_0^u \frac{B(u)}{A(u)}du\right\}\int_0^u \frac{dv}{A(v)} \exp\left\{\int_0^v \frac{B(t)}{A(t)}dt\right\}. \tag{11}$$

STEADY-STATE QUASI-STATIONARY SOLUTION

We have noted earlier that only the particles whose energy exceeds the injection energy $\epsilon_{in} = kT/\sqrt{2}\ \alpha_0$ are accelerated. The number of such particles in the case of a weak acceleration mechanism, i.e., at sufficiently small α_0, is very small. The state of the main mass of the particles, whose energy is less than the injection energy ϵ_{in}, is almost stationary. In fact, were it not for the escape ($\alpha_0 \to 0$, that is, $\epsilon_{in} \to \infty$), this would be in general a stationary state [see Eq. (9)]. Since $\alpha_0 \neq 0$, the energy of certain particles increases to $\epsilon_{in} = kT/\sqrt{2}\ \alpha_0$, and they leave the main region. At sufficiently small α_0, this flux of escaping particles is naturally very weak [see Eq. (3)]. Therefore, although the main state does change with time, its change is very slow, quasi-stationary. The flux of particles from the main region to the acceleration region also changes slowly.*

It was also indicated in Sec. 1 that the influence of a weak acceleration mechanism on particles with energies close to thermal is insignificant. Therefore, in solving Eq. (4) for the case of a weak Fermi acceleration mechanism (6), it is convenient to single out first the region (I) in which the velocities are $u \sim 1$ ($0 \le u \le u_1$), and where the effect of the acceleration mechanism can be neglected in first approximation. In region I, Eq. (4) becomes

$$\frac{\partial f^I}{\partial \tau} - \frac{1}{u^2}\frac{\partial}{\partial u}\left(\frac{1}{u}\frac{\partial f^I}{\partial u} + f^I\right) = 0. \tag{12}$$

The boundary u_1 is subject to the condition that the acceleration mechanism be weak when $u \le u_1$, i.e., $\alpha_0 u_1^3 \ll 1/u_1$. On the other hand, it is necessary that the number of particles which are not in the mean region be negligibly small, i.e., $u_1^3 \exp(-u_1^2/2) \ll 1$. At sufficiently small α_0 (if $\alpha_0^{-3/4} \exp\{-1/2\sqrt{\alpha_0}\} \ll 1$), both conditions can be satisfied.

It must also be considered that escape causes the total number of particles in the main region I to be variable, i.e., although the flux $S = u^{-1}\partial f^I/\partial u + f^I$ vanishes on the boundary $u = 0$ (there are no sources), the flux of particles through the boundary $u = u_1$ is different from zero, $S = -S_0$. At sufficiently small α_0, as indicated above, this flux should be weak ($S_0 \ll N_0$), so that the distribution function in the main region should be close to stationary. As a result, it is natural to seek the solution of Eq. (4) by the method of successive approximations, $f^I = f_1^I + f_2^I + \ldots$, neglecting in first approximation the time variation of f^I, i.e., assuming it to be quasi-stationary. We then obtain instead of (4) the following system of equations for f_1^I, f_2^I, ...

$$\frac{1}{u^2}\frac{\partial}{\partial u}\left(\frac{1}{u}\frac{\partial f_1^I}{\partial u} + f_1^I\right) = 0, \qquad \frac{1}{u^2}\frac{\partial}{\partial u}\left(\frac{1}{u}\frac{\partial f_2^I}{\partial u} + f_2^I\right) = \frac{\partial f_1^I}{\partial \tau}, \ldots \tag{13}$$

From the first equation of (13) we find that, in first approximation, the distribution function in the main region is Maxwellian:

$$f_1^I = C^I \exp\{-u^2/2\}, \tag{14}$$

as should be. The constant C^I is determined from the normalization condition

$$N = \int_0^{u_1} f_1^I u^2\,du = C^I \int_0^{u_1} u^2 \exp\{-u^2/2\}\,du,$$

where N is the total number of particles with velocity $u \le u_1$.* Since we have assumed that u_1 is sufficiently large, so that terms of order $u_1^3 \exp(-u_1^2/2)$ can be neglected, we get

*The time variation of the distribution function in the case of the weak acceleration mechanism considered here is so to speak analogous to the leakage of water from a large reservoir through a very small hole. It is clear that all the water will leak out after an infinite time, i.e., there is no perfectly stationary state, but the level of the water in the reservoir changes very slowly. The escape flux itself soon reaches a definite value, which can subsequently change only slowly (quasi-stationarily) with a change in the overall level of water in the reservoir.

*Since the system under consideration is supposed to be homogeneous in coordinate space, N can be taken to mean both the number of particles in the medium and their density in coordinate space.

$$C^{\mathrm{I}} = \sqrt{\frac{2}{\pi}}\, N = \sqrt{\frac{2}{\pi}} \Big(N_0 - \int_0^{\tau} S_0 \, d\tau \Big). \qquad (14a)$$

Here N_0 is the total number of particles in the system, and S_0 is the flux of particles escaping from the main region I through the surface $u = u_1$.

In the next higher approximation we obtain

$$\frac{1}{u^2} \frac{\partial}{\partial u} \left\{ \frac{1}{u} \frac{\partial f_2^{\mathrm{I}}}{\partial u} + f_2^{\mathrm{I}} \right\} = - S_0 \sqrt{\frac{2}{\pi}} \exp \left\{ -\frac{u^2}{2} \right\}. \qquad (15)$$

Let us integrate this equation, recalling that on the boundary $u = 0$ the flux $S = u^{-1} \partial f_2^{\mathrm{I}} / \partial u + f_2^{\mathrm{I}}$ vanishes. We then have

$$S = - S_0 \sqrt{\frac{2}{\pi}} \int_0^u v^2 \exp \left\{ -\frac{v^2}{2} \right\} dv,$$

i.e., the flux S increases (in absolute value) with increasing u from 0 when $u = 0$ to S_0 when $u = u_1$, as should be. It is interesting that when $u > 1$ the flux rapidly approaches S_0 and remains practically unchanged with increasing u. The flux is thus independent of the choice of the boundary u_1 (provided $u_1 \gg 1$).

Integrating further Eq. (15), we readily obtain also the function f_2^{I}

$$f_2^{\mathrm{I}} = C_2 e^{-u^2/2} - \sqrt{\frac{2}{\pi}} S_0 \int_0^u t^2 e^{-t^2/2} dt + \sqrt{\frac{2}{\pi}} S_0 \frac{u^3}{3} e^{-u^2/2}. \qquad (16)$$

The constant C_2 is determined here by the normalization condition $\int_0^u f_2^{\mathrm{I}} u^2 du = 0$. It is important to note that at sufficiently large $u \gg 1$ the function f_2^{I} tends to a constant value, independent of u: $f_2^{\mathrm{I}} \to -S_0$. In particular, on the boundary of the region under consideration we have $f_2^{\mathrm{I}}(u_1) = -S_0$.

In region II ($u_1 \le u \le u_2$) the influence of the acceleration mechanism must already be accounted for. It is important, however, that through this region passes a constant particle flux $-S_0$, which is independent of u (as shown above, the flux is produced in the region $u \sim 1$ and is practically independent of u when $u \gg 1$). Solving Eq. (4) in region II by the same method as in region I, we find that the distribution function in region II is of the form [cf. Eq. (8)]

$$f^{\mathrm{II}} = C^{\mathrm{II}} \exp \left\{ -\int_0^u \frac{u\, du}{1 + \alpha_0 u^4} \right\} - S_0. \qquad (17)$$

The constant C^{II} is determined from the condition of continuity of the distribution function on the boundary $f^{\mathrm{I}}(u_1) = f^{\mathrm{II}}(u_1)$, i.e.,

$$\sqrt{\frac{2}{\pi}} N \exp \left\{ -\frac{u_1^2}{2} \right\} - S_0 = C^{\mathrm{II}} \exp \left\{ -\int_0^u \frac{u\, du}{1 + \alpha_0 u^4} \right\} - S_0. \qquad (18)$$

Therefore, considering that $\alpha_0 u^4 \ll 1$, we get $C^{\mathrm{II}} = N\sqrt{2\pi}$. Thus, the steady-state solution in region II is

$$f^{\mathrm{II}}(u) = \sqrt{\frac{2}{\pi}} N \exp \left\{ -\int_0^u \frac{u\, du}{1 + \alpha_0 u^4} \right\} - S_0. \qquad (17a)$$

It is important here that as u increases $f^{\mathrm{II}}(u)$ tends to a constant value independent of u,

$$f^{\mathrm{II}}(\infty) = \sqrt{\frac{2}{\pi}} N \exp \left\{ -\int_0^\infty \frac{u\, du}{1 + \alpha_0 u^4} \right\} - S_0 = \sqrt{\frac{2}{\pi}} N \exp \left\{ -\frac{\pi}{4\sqrt{\alpha_0}} \right\} - S_0. \qquad (19)$$

We have thus determined the distribution functions in regions I and II. In the case of a weak flux $S_0 \ll N$, the resulting expression (14)—(17) is almost stationary, and changes with time only to the extent that the number of particles changes in the main region, $N = N_0 - \int_0^\tau S_0 d\tau$. The only unknown in (14)—(17) is the flux S_0, which can be determined from the normalization condition. In fact, the total number of particles passing from region I to region II during the time τ is $\int_0^\tau S_0 d\tau$. The total change in the normalization integral of the distribution function over the entire region $u > u_1$ should obviously be equal to this quantity,

$$\Delta N = \int_{u_1}^{\infty} u^2 [f(u, \tau) - f(u, 0)]\, du = \int_0^\tau S_0 \, d\tau, \qquad (20)$$

where $f(u, 0)$ is the initial (Maxwellian) distribution function. From this we can obtain the flux S_0. We first consider that the quasi-stationary distribution (17a) is valid up to the boundary $u = u_2$. The boundary itself, naturally, moves with increasing τ towards higher values of u, at which the quasi-stationary distribution function f^{II} becomes a constant quantity, $f^{\mathrm{II}}(\infty)$ [see Eq. (19)]. Consequently, at sufficiently large τ, i.e., at sufficiently large u_2, the number of particles in the stationary region II alone (ΔN^{II}) is given by

$$\Delta N^{\mathrm{II}} = \int_{u_1}^{u_2} f^{\mathrm{II}} u^2 du \gtrsim \int^{u_2} f^{\mathrm{II}}(\infty) u^2 du = f^{\mathrm{II}}(\infty) u_2^3/3. \qquad (20a)$$

We now determine the time variation of the boundary of the steady-state quasi-stationary distribution u_2. In the region of large u ($u > \sqrt{\epsilon_{in}/kT} \sim 1/\sqrt{\alpha_0}$) the collisions are insignificant and the distribution function changes only because of the effect of the acceleration mechanism on the particle. Equation (4) obviously assumes in this region the form

$$\frac{\partial f}{\partial \tau} - \frac{1}{u^2}\frac{\partial}{\partial u}\left(\alpha_0 u^3 \frac{\partial f}{\partial u}\right) = 0. \tag{21}$$

The boundary u_2 of the quasi-stationary region obviously changes with time like the average velocity $\bar{u} = 1/\Delta N \int_{u_1}^{\infty} u^3 f(u, \tau)\,du$. From (21) we obtain for $\bar{u}$ the equation $d\bar{u}/d\tau = 3\alpha_0$ or $\bar{u} = \bar{u}_0 + 3\alpha_0\tau$. Consequently, u_2 also increases with time in proportion to τ:

$$u_2 = u_{20} + 3\alpha_0\tau, \tag{22}$$

where $u_{20} \sim \sqrt{\epsilon_{in}/kT} \sim 1/\sqrt{\alpha_0}$. Substituting now (22) in (20a), we find that for sufficiently large $\tau\,(\tau \gg \alpha_0^{-3/2})$

$$\Delta N^{II} \approx f^{II}(\infty)\, u_2^3/3 = 9\alpha_0^3\tau^3 f^{II}(\infty).$$

We now assume that the number of particles in the quasi-stationary region II $(u_1 \le u \le u_2)$ in no case exceeds ΔN, the total number of particles in the entire region $u \ge u_1$, i.e.,

$$\Delta N^{II} \approx 9\alpha_0^3\tau^3 f^{II}(\infty) \leqslant \Delta N = \int_0^{\tau} S_0\, d\tau.$$

Since the steady-state flux S_0 is almost stationary (it can only decrease slowly with increasing τ), it follows from this inequality that

$$f^{II}(\infty) \leqslant \bar{S}_0 (3\alpha_0^{3/2}\tau)^{-2},$$

where $\bar{S}_0$ is the average value of S_0. It is clear therefore that the steady-state value is $f^{II}(\infty) = 0$ (time to establish steady state $\Delta\tau \gtrsim \alpha_0^{-3/2}$):

$$f^{II}(\infty) = \sqrt{\frac{2}{\pi}}\, N \exp\left\{-\int_0^{\infty} \frac{u\,du}{1+\alpha_0 u^4}\right\} - S_0 = 0.$$

Consequently

$$S_0 = \sqrt{\frac{2}{\pi}}\, N \exp\left\{-\int_0^{\infty} \frac{u\,du}{1+\alpha_0 u^4}\right\} = \sqrt{\frac{2}{\pi}}\, N \exp\left\{-\frac{\pi}{4\sqrt{\alpha_0}}\right\}. \tag{23}$$

The ion density N is determined here by the equation

$$dN/dt = -S_0\nu_0 = -\nu_0 N \exp\{-\pi/4\sqrt{\alpha_0}\},$$

in the solution of which, naturally, it is necessary to take into account the fact that ν_0 and α_0 depend on N.

Thus, in the system considered of identical particles interacting in accordance with a Coulomb law and under the influence of a weak $(\alpha_0 \ll 1)$ Fermi acceleration mechanism, the quasi-stationary distribution established after a time

$$\Delta\tau \gg \alpha_0^{-3/2} \tag{24}$$

is*

$$f(u, \tau) = \sqrt{\frac{2}{\pi}}\, N(\tau)\left[\exp\left\{-\int_0^{u} \frac{u\,du}{1+\alpha_0 u^4}\right\} - \exp\left\{-\int_0^{\infty} \frac{u\,du}{1+\alpha_0 u^4}\right\}\right]. \tag{25}$$

The condition for the existence of a quasi-stationary distribution (25) is written here in the form $S_0\Delta\tau \ll N$, or

$$\alpha_0^{-3/2} \exp\{-\pi/4\sqrt{\alpha_0}\} \ll 1, \tag{26}$$

This condition is always satisfied at sufficiently small $\alpha_0\,(\alpha_0 \ll 10^{-2})$. It is easy to see that the other requirements stipulated above during the solution process are also satisfied if conditions (24) and (26) are satisfied.

It is interesting to note that the distribution function (25) differs from the stationary distribution function (9) only by a constant (second term in the square brackets), which determines precisely the value of the flux of escaping particles. The distribution function (25), unlike the stationary distribution function (9), vanishes as $u \to \infty$.

FLUX OF ESCAPING PARTICLES

The steady-state flux of the escaping particles, as is clear from (23), is

$$dN/dt = \sqrt{2/\pi}\, N\nu_0 \exp\{-\pi/4\sqrt{\alpha_0}\}. \tag{23a}$$

Comparing this exact expression for the first flux with the "elementary" formula (3), obtained under the assumption that the distribution function is Maxwellian up to the injection energy,

$$dN/dt \sim N\nu \exp\{-\varepsilon_{in}/kT\} = N\nu \exp\{-1/\sqrt{2}\,\alpha_0\}, \tag{3b}$$

we see that the difference between them is very great. It is important that this difference is not only quantitative but also qualitative (a different dependence on the parameter α_0 which characterizes the acceleration mechanism). The reason is that the region in which the acceleration mech-

*Strictly speaking, the distribution (14)–(16) is valid in region I, but in considering the distribution function itself (and not the flux) we can neglect the difference between the distribution (14)–(16) and (25). It must also be noted that condition (24) indicates only the time required to establish quasi-stationary distribution (25), up to injection energies $u \sim \alpha_0^{-1/2}$. The time to establish the distribution (25) is accordingly greater at higher energies: $\Delta\tau \sim u^3 \gg \alpha_0^{-3/2}$.

anism begins to influence substantially the form of the distribution function is much closer to the injection energy, and consequently the distribution function at $\epsilon \approx \epsilon_{in}$ is much different from Maxwellian. In fact, the region in which the substantial deviation of the distribution function from Maxwellian begins is determined, as is clear from (9) and (25), by the condition $\alpha_0 u^4 \sim 1$ or $\epsilon \sim \epsilon_{in}/\sqrt{\alpha_0} \ll \epsilon_{in}$. From a comparison of (3a) with the exact formula (23a) it is seen that actually the elementary formula is never applicable.

The solution obtained for the special form of acceleration mechanism (6) can be readily extended to cover more general cases. In particular, if the equation for the distribution function has the general form (10), then the expression for the steady-state flux (at a weak acceleration mechanism) has the form

$$S_0 = \sqrt{2/\pi}\, N \left[\int_0^\infty \frac{du}{A(u)} \exp\left\{ \int_0^u \frac{B(v)}{A(v)} \quad \right\} \right] . \qquad (27)$$

In particular, if $B(u) = 1$, then

$$S_0 = \sqrt{\frac{2}{\pi}}\, N \exp\left\{ -\int_0^\infty \frac{du}{A(u)} \right\} . \qquad (27a)$$

When $A(u)$ has a value $1/u + \alpha_0 u^3$, formula (27a) naturally coincides with (23). The distribution function in this general case is

$$f(u, \tau) = \sqrt{\frac{2}{\pi}}\, N(\tau) \exp\left\{ -\int_0^u \frac{B(u)}{A(u)}\, du \right\} \left[1 - \int_0^u \frac{dv}{A(v)} \right.$$
$$\left. \times \exp\left\{ \int_0^v \frac{B(t)}{A(t)}\, dt \right\} \Big/ \int_0^\infty \frac{dv}{A(v)} \exp\left\{ \int_0^v \frac{B(t)}{A(t)}\, dt \right\} \right] . \qquad (28)$$

It is seen from (27) that one can distinguish two essentially different cases in which the flux of escaping particles is different from zero: either $A(u)$ increases much faster than $B(u)$ as $u \to \infty$ (as in the example considered above), or the function $B(u)$ vanishes at a certain u_c and subsequently becomes negative.* In the latter case, for a weak acceleration mechanism, u_c is always much greater than unity. It is consequently possible to carry out the integration in (27) (by the saddle-point method). We then find that

$$S_0 = \frac{1}{\pi} N \left(B \frac{dA}{du} - A \frac{dB}{du} \right)^{1/2}_{u=u_c} \exp\left\{ -\int_0^{u_c} \frac{B(u)}{A(u)}\, du \right\} , \qquad (27b)$$

where the velocity u_c is determined by the condition $B(u_c) = 0$.

3. A weak Fermi acceleration mechanism in a plasma changes significantly only the velocity distribution of the ions. The acceleration mechanism hardly affects the electron distribution function, which can therefore be considered Maxwellian. The equation for the ion distribution function, with allowance for interactions between the ions and with the electrons, has the form

$$\frac{\partial f}{\partial \tau} - \frac{1}{u^2} \frac{\partial}{\partial u} \left\{ \left(\frac{1}{u} + \frac{1}{u} \gamma^2 G\left(\frac{u}{\sqrt{2}\gamma} \right) \right.\right.$$
$$\left.\left. + u^2 \alpha(u) \right) \frac{\partial f}{\partial u} + \left(1 + \gamma^2 G\left(\frac{u}{\sqrt{2}\gamma} \right) \right) f \right\} = 0. \qquad (29)$$

Here $u = v/\sqrt{kT/M}$ and $\tau = \nu_{0i} t$, where T is the plasma temperature and ν_{0i} is the collision frequency for the ions (1), assuming single collisions for simplicity. The terms containing γ describe the interaction between the ions and the electrons. Here $\gamma^2 = M/m$ (m is the electron mass) and

$$G(x) = \frac{2}{\sqrt{\pi}} \left(\int_0^x e^{-z^2} dz - x e^{-x^2} \right) . \qquad (29a)$$

At small $x \ll 1$ we have $G(x) = 4x^3/3\sqrt{\pi}$; at large $x \gg 1$ we have $G(x) = 1 - 2xe^{-x^2}/\sqrt{\pi}$. Values of the function $G(x)$ for $x \sim 1$ are given, for example, in reference 2. Finally, the term with $\alpha(u)$, as above, describes the effect of the acceleration mechanism on the ions

$$\alpha(u) = \alpha_0 u^\lambda, \qquad \alpha_0 = {}^2/_3 v_c^2 / \nu_0 L\left(\sqrt{kT/M}\right) \sqrt{kT/M}. \qquad (30)$$

All the quantities here are the same as in Sec. 2, except that a somewhat more general expression than (6) is used for $\alpha(u)$, making it possible to describe not only the case of a constant ion mean free path between two collisions with clouds, but also the case of an arbitrary power-law dependence of L on the ion velocity v: $L(v) = Q_0 v^{1-\lambda}$

Using (23) and (27) for the steady-state flux of the escaping ions, we find that in our case

$$\frac{dN}{dt} = \nu_0 S_0 = \sqrt{\frac{2}{\pi}}\, N \nu_0 \left[\int_0^\infty \frac{u\, du}{1 + \gamma^2 G(u/\sqrt{2}\gamma) + \alpha_0 u^{\lambda+3}} \right.$$
$$\left. \times \exp\left\{ \int_0^u \frac{-t\,[1 + \gamma^2 G(t/\sqrt{2}\gamma)]}{1 + \gamma^2 G(t/\sqrt{2}\gamma) + \alpha_0 t^{\lambda+3}} \right\} \right]^{-1} . \qquad (31)$$

Considering that the principal role is played in this expression for dN/dt by the exponential term, we can, integrating by parts, convert this term to

*This case is usually characteristic of guided acceleration mechanisms (for example, acceleration by means of an electric field, acceleration of particles in a plasma contained between two contracting walls, etc.). It should be noted that although one cannot assume in the analysis of guided acceleration mechanisms, as was done above, that the distribution function depends only on the absolute value of the velocity, nevertheless expressions of the type (27b) for the flux of the escaping particles hold in this case, too.

$$\frac{dN}{dt}=\sqrt{\frac{2}{\pi}}N\nu_0(1+\gamma^2\bar{G})\exp\left\{-\int_0^\infty\frac{u\,du\,(1+\gamma^2G(u/\sqrt{2}\gamma))}{1+\gamma^2G(u/\sqrt{2}\gamma)+\alpha_0u^{\lambda+3}}\right\}, \tag{32}$$

where $\bar{G}$ is a certain average value of the function $G(u/\gamma\sqrt{2})$; we shall show below that $\bar{G}$ is always equal to unity in the cases of interest to us.

Our principal problem is to analyze the form of the function

$$J(\gamma,\alpha_0,\lambda)=\int_0^\infty\frac{u\,du\,(1+\gamma^2G(u/\sqrt{2}\gamma))}{1+\gamma^2G(u/\sqrt{2}\gamma)+\alpha_0u^{\lambda+3}}$$

$$=2\gamma^2\int_0^\infty\frac{x\,dx\,(1+\gamma^2G(x))}{1+\gamma^2G(x)+2^{(\lambda+3)/2}\alpha_0\gamma^{\lambda+3}x^{\lambda+3}}. \tag{33}$$

We note first that when $\lambda\leq-1$ the function $J(\gamma,\alpha_0,\lambda)$ becomes infinite. This means that the flux of escaping ions is zero when $\lambda\leq-1$. In other words, in this case the distribution function is stationary, as it should be (see Sec. 2). We shall assume $\gamma^2=M/m$ always to be a very large quantity. It is easy to see that we can consequently always neglect unity compared with the other terms in (33),* i.e., we can rewrite (33) as

$$J=2\gamma^2\int_0^\infty\frac{xG(x)\,dx}{G(x)+2^{(\lambda+3)/2}\alpha_0\gamma^{\lambda+1}x^{\lambda+3}}. \tag{33a}$$

It is now easy to obtain the essential limiting values of the function J, namely, when $p=2^{(\lambda+3)/2}\alpha_0\gamma^{\lambda+1}\gg1$,

$$J=\frac{2\gamma^2}{p}\int_0^\infty\frac{G(x)}{x^{\lambda+2}}dx=\frac{\gamma^{1-\lambda}}{\alpha_0}2^{-(1+\lambda)/2}q(\lambda),\qquad q(\lambda)=\int_0^\infty\frac{G(x)}{x^{\lambda+2}}dx, \tag{34}$$

where $\lambda<2$. The numerical values of $q(\lambda)$ are listed in the table. In the second limiting case, when $p\ll1$,

$$J=\frac{2\gamma^2}{p^{2/(\lambda+3)}}r(\lambda)=\left(\frac{\gamma^2}{\alpha_0}\right)^{2/(\lambda+3)}r(\lambda).$$

The values of the function $r(\lambda)$ are also listed in the table, from which it is seen that the functions

λ	$q(\lambda)$	$r(\lambda)$
−1.0	∞	∞
−0.5	2.04	2.07
0	1.13	1.21
0.5	0.93	0.92
1.0	1.0	0.78

*In other words, the basic decisive influence on the flux of the escaping ions is exerted by their interaction with the electrons and not with each other. This fact agrees with the well-known circumstance that the main energy losses of fast ions in a plasma are due to their interactions with electrons, and not with ions.

$q(\lambda)$ and $r(\lambda)$ differ little from each other. As λ changes from −0.5 to 1, these functions change very little, and remain in fact close to unity.

It must be emphasized that only the first of the foregoing limiting cases is of prime significance. In fact, as can be seen from (32), the flux is proportional to e^{-J}, and when $p\lesssim1$ the function J is greater than γ^2, where $\gamma^2=M/m$ is always a very large quantity, on the order of 10^3 to 10^5. Consequently, only the first limiting case can be of practical interest, when $p\gg1$ and the flux S_0 assumes sensible values at sufficiently large p. It is also easy to verify that $\bar{G}$, the mean value of $G(x)$ in the integral (32), is always close to unity.* Therefore, when calculating the flux of the escaping ions, we can always use the simple expression (34):

$$\frac{dN}{dt}=\sqrt{\frac{2}{\pi}}N\nu_0\gamma^2\exp\left\{-2^{-(1+\lambda)/2}q(\lambda)\frac{\gamma^{1-\lambda}}{\alpha_0}\right\}$$

$$=\sqrt{\frac{2}{\pi}}N\nu_0\frac{M}{m}\exp\left\{-C_0\frac{NT^{-(1+\lambda)/2}}{v_{06}^2M}\right\}, \tag{35}$$

$$C_0=3\pi e^4(2m)^{(1-\lambda)/2}q(\lambda)/Q_0k^{(\lambda+1)/2}\ln(kTD/e^2). \tag{35a}$$

It is seen therefore that if the cloud velocity v_c^2 is independent of the plasma parameters, the flux diminishes exponentially with increasing plasma density and increases with increasing plasma temperature.

It is also interesting that the flux of the escaping ions increases rapidly with increasing ion mass. It must be emphasized here, to be sure, that only singly-charged ions are considered. The flux of Z-fold ions is given by the same formulas, (35) and (35a), but the exponential term must be multiplied by Z^2N_e/N_i, and the term preceding the exponential must be multiplied by N_e/Z^2N_i.† It is clear therefore that the ion flux diminishes rapidly with increasing ion charge: the exponential term is proportional to Z^2 (if the number of multiply-charged ions is relatively small, so that $N_e\approx N_i$), or even to Z^3 (if all the ions are Z-fold ionized so that $N_e=ZN_i$).

*In fact, as is clear from (31), $\bar{G}$ is determined by that region of the values of G(x), in which the quantity

$$\frac{\gamma^2G(x)\,x}{G(x)+px^{\lambda+3}}\exp\left\{2\gamma^2\int_0^x xG(x)/(G(x)+px^{\lambda+3})dx\right\}$$

has a maximum. This quantity always has a maximum at large x ($x_{max}\gg1$). Therefore $\bar{G}=G(x_{max})=1$ [the function G(x) is close to unity for large x; see (29a)].

†We note also that the frequency of collisions between ions, ν_{0i}, is proportional to Z^4 and the parameter $\tau=t\nu_{0i}$ changes accordingly.

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