A RELATIVISTIC TRANSPORT EQUATION FOR A PLASMA. II*

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We use the chain of equations for the relativistic distribution functions which we obtained earlier[1] to derive a relativistic transport equation in second approximation for a plasma. We derive first a transport equation in which only the retarded interaction of charged particles is taken into account. This equation is in a particular case the same as the one obtained in a paper by Belyaev and Budker.[2] We consider a relativistic Fokker-Planck equation for a plasma in which the retarded interaction between charged particles and the excitation of plasma oscillations by non-equilibrium charged particles is taken into account.

1. THE SECOND-APPROXIMATION TRANSPORT EQUATION FOR A RELATIVISTIC PLASMA

We use in this section the results of reference 1 to derive a second-approximation classical transport equation for a plasma; this is the relativistic analogue of the equation obtained by Bogolyubov. Belyaev and Budker[2] considered as a special case of this the equation where radiation and pair production were neglected. It becomes the same as Landau's equation as $c \to \infty$.

We first derive the transport equation neglecting radiation. We can obtain this, for instance, as follows.

We start from Eqs. (12) and (13) of reference 1 for the random functions $N_{q_i p_i}$ and F_{ik}, with this difference, however, that we take instead of Eqs. (13) for the tensor F_{ik} the equations for the four-potential A_i

$$u_i \partial N_{q_i p_i} / \partial q_i + \frac{e}{c} F_{ik} u_k \partial N_{q_i p_i} / \partial p_i = 0, \tag{1}$$

$$\nabla^2 A_i - \frac{1}{c^2} \partial^2 A_i / \partial t^2 = -\frac{4\pi e}{c} \int u_i N_{q_i p_i} d^4 p, \quad \partial A_i / \partial q_i = 0. \tag{2}$$

We assume that there is a positive-charge background of ions. For the sake of simplicity, we shall write sometimes N_{x_i} instead of $N_{q_i p_i}$, where x_i indicates the eight-vector (q_i, p_i).

As we intend to obtain an approximate transport equation neglecting radiation, we can write the solution of Eq. (2) in the form

$$A_i(q_i) = \frac{ec}{2\pi^2} \iint_{t_0}^{t} u_i' N_{q_i' p_i'} e^{-i\mathbf{k}(\mathbf{q}-\mathbf{q}')} \omega_k^{-1} \sin \omega_k (t - t')\, d^3 k\, d\Omega' \tag{3}$$

We can let the lower limit tend to $-\infty$, provided the time needed to establish statistical equilibrium τ and the characteristic dimensions l of the system are such that $1/c \ll \tau$. $d\Omega' = d^4q' d^4p'$ in Eq. (3).

Using Eq. (3) to eliminate the potential A_i from Eq. (1) and writing

$$L_{il} = \frac{e^2}{2\pi^2} \int \left\{ \frac{\partial}{\partial q_i} (u_l' e^{-i\mathbf{k}\mathbf{q}} \omega_k^{-1} \sin \omega_k t) - \frac{\partial}{\partial q_l} (u_i' e^{-i\mathbf{k}\mathbf{q}} \omega_k^{-1} \sin \omega_k t) \right\} d^3 k, \tag{4}$$

we get the following equation for the function $N_{q_i p_i}$

$$u_i \partial N_{q_i p_i} / \partial q_i + \iint_{t_0}^{t} L_{il} (q_i - q_i', p_i') N_{q_i' p_i'} d\Omega' u_l \partial N_{q_i p_i} / \partial p_i = 0. \tag{5}$$

Using (5) we can find a chain of equations for the moments of the random function $N_{q_i p_i}$. We get the first equation for $\overline{N_{q_i p_i}}$ by a straight averaging of (5). To obtain the equation for the second moment $\overline{N_{x_i} N_{x_i'}}$ for $t \neq t'$ and $q \neq q'$, we multiply (5) by $N_{x_i'}$ and average

$$u_i \partial \overline{N_{x_i} N_{x_i'}} / \partial q_i + \iint_{t_0}^{t} L_{il} u_l \frac{\partial}{\partial p_i} (\overline{N_{x_i} N_{x_i'} N_{x_i''}})\, d\Omega'' = 0, \tag{6}$$

where $d\Omega'' = d^4q'' d^4p''$. By interchanging the variables $q_i \rightleftarrows q_i'$ and $p_i \rightleftarrows p_i'$ we can obtain another equation for the second moment which is the same as the first.

It is sometimes more convenient to use equations for the corresponding distribution functions instead of the chain of equations for the moments of the random functions $N_{q_i p_i}$. We have given in reference 1 the equations which connect the mo-

*The present paper is a direct continuation of reference 1 by the author.

ments of the random functions $N_{\mathbf{qp}}$ with the distribution functions for the variables $\mathbf{q}$, $\mathbf{p}$, and t. One can write down the corresponding equations for the functions $N_{q_i p_i}$ of eight variables. For instance,

$$\overline{N_{x_i} N_{x_i'}} = N(N-1) f_2 + N \int \delta(x_i - x_i'(s'))\, ds' f_1(x_i)$$

and similar equations for higher moments.

Subsituting these equations into the equations for the moments, and taking into account that $L_{il}(q_i, q_i) = 0$, we get the following two equations as the first of a chain of equations for relativistic distribution functions

$$u_i \frac{\partial f_1}{\partial q_i} + N \iint\limits_{t_0}^{t} L_{il} u_l \frac{\partial f_2}{\partial p_i} d\Omega' = 0, \qquad (7)$$

$$u_i \frac{\partial f_2}{\partial q_i} + \int\limits_{t_0}^{t} \int L_{il} u_l \frac{\partial f_2}{\partial p_i} \delta(x_i'' - x_i'(s'))\, d\Omega'' ds'$$
$$+ N \int\limits_{t_0}^{t} \int L_{il} u_l \frac{\partial f_3}{\partial p_i} d\Omega'' = 0, \quad t \neq t' \qquad (8)$$

and a similar second equation for f_2.

We denote by μ the parameter that characterizes the weakness of the interaction. Let $\mu \sim e$. We cut off the chain of Eqs. (7) and (8) by putting

$$f_2 = f_1 f_1 + \mu^2 g, \quad f_3 = f_1 f_1 f_1. \qquad (9)$$

Here $g = g(x_i, x_i')$ is the relativistic correlation function.

The approximate set of equations for f_1 and g becomes, after integration over q_i'' and p_i'',

$$u_i \frac{\partial f_1}{\partial q_i} + N \int\limits_{t_0}^{t} \int L_{il} u_l f_1 d\Omega' \frac{\partial f_1}{\partial p_i} + N \iint\limits_{t_0}^{t} L_{il} u_l \frac{\partial}{\partial p_i} g\, d\Omega' = 0, \qquad (10)$$

$$u_i \frac{\partial}{\partial q_i} g + \int\limits_{t_0}^{t} L_{il} u_l \frac{\partial f_1}{\partial p_i} f_1(q_i'(s'), p_i'(s'))\, ds' = 0. \qquad (11)$$

The correlation function which determines the connection between different positions of particles at different times enters into Eq. (10) for f_1. It follows from Eq. (11) that the correlation function itself is determined by the distribution function of the first particle at time t and by the distribution function of the second particle at all times, within the limits $t \geq t' \geq t_0$, i.e., by the complete motion of the second (primed) particle in that time interval. The dependence of the right-hand side of the equation on q_i manifests itself in that the second particle is during its motion at the point q' in the time t'.

In Eq. (11) for g we take thus only the influence of the motion of the second particle on the first one into account. The reaction of the first particle on the second one is described by the correlation function satisfying the second equation. This one is obtained from Eq. (11) by the substitution $x_i \rightleftarrows x_i'$ and has the following form

$$u_i' \frac{\partial}{\partial q_i} g + \int\limits_{t_0}^{t} L_{il}(q_i' - q_i, p_i) u_l' \frac{\partial f_1}{\partial p_i'} f_1(q_i(s), p_i(s))\, ds = 0. \qquad (11')$$

The distribution function of the first particle at times $t_0 \leq t \leq t'$ occurs in the right-hand side of Eq. (11). Because the particles are identical, we must substitute into the right-hand side of Eq. (10) a symmetrical combination of solutions of Eqs. (11) and (11′).

We write Eq. (11) explicitly, substituting into it Eq. (4) for L_{il}

$$u_i \frac{\partial g}{\partial q_i} = -\frac{e^2}{2\pi^2} \iint\limits_{t_0}^{t} \Big\{ \frac{\partial}{\partial q_i} (v_l' e^{-i\mathbf{k}(\mathbf{q}-\mathbf{q}')} \omega_k^{-1} \sin \omega_k (t-t'))$$
$$- \frac{\partial}{\partial q_l} (v_i' e^{-i\mathbf{k}(\mathbf{q}-\mathbf{q}')} \omega_k^{-1} \sin \omega_k (t-t')) \Big\}$$
$$\times u_l \frac{\partial f_1}{\partial p_i} f_1(q_i'(\tau), p_i'(\tau))\, d\tau d^3k. \qquad (12)$$

In this equation $v_k' = u_k'/\gamma'$, $d\tau' = ds'\gamma'$, $q' = q'(\tau)$, $t' = t'(\tau) = \tau$, and $\gamma' = \epsilon'/m_0c^2$. There is a similar equation for the second correlation function.

As the interaction is weak, one can determine the trajectory of the second particle in the right-hand side of Eq. (12) by the method of successive approximations. As we wish to obtain the transport equation up to terms of order μ^4, we must restrict ourselves to the zeroth approximation when solving the equations of motion. In that approximation $\mathbf{q}(\tau) = \mathbf{v}\tau + \mathbf{C}$ and $\mathbf{q}'(\tau) = \mathbf{v}'\tau + \mathbf{C}'$. We determine the constants $\mathbf{C}$ and $\mathbf{C}'$ from the conditions $\mathbf{q}(\tau) = \mathbf{q}(t) \equiv \mathbf{q}$ at $\tau = t$, and $\mathbf{q}'(\tau) = \mathbf{q}'(t') \equiv \mathbf{q}'$ at $\tau = t'$. We therefore have $\mathbf{C} = \mathbf{q} - \mathbf{v}t$ and $\mathbf{C}' = \mathbf{q}' - \mathbf{v}'t'$. We get thus

$$\mathbf{q}(\tau) = \mathbf{v}(\tau - t) + \mathbf{q}, \quad \mathbf{q}'(\tau) = \mathbf{v}'(\tau - t') + \mathbf{q}'. \qquad (13)$$

Using Eqs. (13) and the fact that

$$f_1(q' - v'(t' - \tau), \tau, p_i') \approx f_1(q', t', p_i'),$$

we can write Eq. (12) as follows

$$u_i \frac{\partial g}{\partial q_i} = -\frac{e^2}{2\pi^2} \int\limits_{t_0}^{t} \int \Big\{ \frac{\partial}{\partial q_i} (v_l \exp\{-i\mathbf{k}(\mathbf{q} - \mathbf{q}' - \mathbf{v}'(\tau - t'))\}$$
$$\times \omega_k^{-1} \sin \omega_k (t - \tau) - \frac{\partial}{\partial q_l} (v_i' \exp\{-i\mathbf{k}$$
$$\times (\mathbf{q} - \mathbf{q}' - \mathbf{v}'(\tau - t'))\}$$
$$\times \omega_k^{-1} \sin \omega_k (t - \tau)) \Big\} u_l \frac{\partial f_1}{\partial p_i} f_1(q_i' p_i')\, d\tau d^3k \qquad (14)$$

and a similar second equation for the function of g.

Making in Eq. (14) a change of variables $t-\tau = t^*$, letting $t_0 \to -\infty$, and integrating over t^*, we get in this case the following equations for g

$$u_i \frac{\partial g}{\partial q_i} = \frac{e^2}{2\pi^2}\int \frac{k'_i v'_l - k'_l v'_i}{\omega_k^2 - (\mathbf{kv}')^2}$$
$$\times \sin \mathbf{k}\,[\mathbf{q}-\mathbf{q}'-\mathbf{v}'(t-t')]\, d^3k u_l \frac{\partial f_1}{\partial p_i} f_1(p'_i),$$
$$u'_i \frac{\partial g}{\partial q'_i} = \frac{e^2}{2\pi^2}\int \frac{k_i v_l - k_l v_i}{\omega_k^2 - (\mathbf{kv})^2}$$
$$\times \sin \mathbf{k}\,[\mathbf{q}'-\mathbf{q}-\mathbf{v}(t'-t)]\, d^3k u'_l \frac{\partial f_1}{\partial p'_i} f_1(p_i). \quad (15)$$

In these equations k_i and k'_i are wave four-vectors with components $\mathbf{k}$, $i(\mathbf{k}\cdot\mathbf{v})/c$ and $\mathbf{k}$, $i(\mathbf{k}\cdot\mathbf{v}')/c$ respectively.

The solution of these equations for time intervals which are larger than the correlation time, so that one can neglect the initial values of the functions g, can be written in the form

$$g = \frac{e^2}{2\pi^2\gamma}\int_0^\infty\int \frac{k'_i v'_l - k'_l v'_i}{\omega_k^2 - (\mathbf{kv}')^2} \sin \mathbf{k}\,[\mathbf{q}-\mathbf{q}'-\mathbf{v}'(t-t')-(\mathbf{v}-\mathbf{v}')\tau]$$
$$\times\, u_l\,(\partial f_1/\partial p_i)_{\mathbf{q}\to\mathbf{q}-\mathbf{v}\tau} f_1(q'_i p'_i)\, d^3k d\tau \quad (16)$$

with a corresponding expression for the second correlation function. When obtaining Eq. (16) we have used the fact that $u_i k'_i = \gamma(\mathbf{k}\cdot\mathbf{v} - \mathbf{k}\cdot\mathbf{v}')$.

Using (16) to eliminate g from Eq. (10), we get the required relativistic transport equation in second approximation. It is the same in the nonrelativistic approximation as Eq. (10.18) of Bogolyubov's paper.[3]

From the second-approximation transport equation obtained here we can obtain the relativistic generalization of the well-known Landau equation. To do this we consider the case of a distribution function which varies so slowly in space-time that one can consider it to be constant over a region defined by the correlation radius and the corresponding correlation time.

In that case we can drop in Eq. (16) the derivatives with respect to the coordinates, i.e., we can assume

$$(\partial f_1/\partial p_i)_{\mathbf{q}\to\mathbf{q}-\mathbf{v}\tau} = \partial f_1/\partial p_i.$$

Substituting the expression for g into Eq. (10), integrating over q'_i, and assuming the function $f_1(q'_i)$ to be independent of q'_i we get the following transport equation

$$u_i \frac{\partial f_1}{\partial q_i} = \frac{\partial}{\partial p_i}\int \mathscr{E}_{il}(p_i, p'_i)\left\{\frac{\partial f_1}{\partial p_l} f_1 - \frac{\partial f_1}{\partial p'_l} f_1\right\} d^4p'. \quad (17)$$

The components of the $\mathscr{E}_{il}$ tensor are defined by the equation

$$\mathscr{E}_{il} = 2e^4 N (u_n u'_n)^2 \int \frac{k_i k_l}{(\omega_k^2 - \omega^2)^2}\,\delta(k_j u_j)\,\delta(k_j u'_j)\, d^3k d\omega. \quad (18)$$

We can write Eq. (18) in a different form

$$\mathscr{E}_{il} = \frac{e^2 N}{32\pi^4 c^2}\int k_i k_l (A_j u_j)^2 d^3k d\omega,$$

where the $A_j(\mathbf{k}, \omega)$ are the Fourier components of the 4-potential produced by the charged particle in the independent-motion approximation.

We can integrate over $\mathbf{k}$ and ω in Eq. (18). If we use the fact that the tensor $\mathscr{E}_{il}$ is symmetric in u_i and u'_i and that $u_i\mathscr{E}_{il} = 0$ and $u'_i\mathscr{E}_{il} = 0$, we need only find the trace of the tensor $\mathscr{E}_{il}$ to determine the $\mathscr{E}_{il}$. As a result we get the following equation

$$\mathscr{E}_{il} = 2\pi e^4 N (u_n u'_n)^2 c^{-5} L\,[(u_n u'_n)^2 c^{-4} - 1]^{-3/2}$$
$$\times \{[c^{-4}(u_n u'_n)^2 - 1]\,\delta_{il} - c^{-2}(u_i u_l + u'_i u'_l)$$
$$- c^{-4}(u_n u'_n)(u_i u'_l + u'_i u_l)\}. \quad (19)$$

$L = \int dk/k$ is the Coulomb logarithm.

Equation (17) is the same as the transport equation obtained in Belyaev and Budker's paper.[2]

It is often more convenient to use a transport equation for the usual distribution function $F_1(\mathbf{q}, \mathbf{p}, t)$. Taking into account the connection between f_1 and F_1 (see references 1 and 2)

$$f_1(p_i) = F_1(\mathbf{q}, \mathbf{p}, t)\,\delta(\varepsilon - c\sqrt{m_0^2c^2 + p^2})\, m_0 c^2/\varepsilon \quad (20)$$

substituting Eq. (20) into Eq. (17), and integrating over ϵ, we get the following equation

$$\frac{\partial F_1}{\partial t} + \mathbf{v}\frac{\partial F_1}{\partial \mathbf{q}}$$
$$= \frac{\partial}{\partial p_\alpha}\int \mathscr{E}^*_{\alpha\beta}\left\{\frac{\partial F_1}{\partial p_\beta} F_1 - \frac{\partial F_1}{\partial p'_\beta} F_1\right\} d^3p'; \quad \alpha, \beta = 1, 2, 3. \quad (21)$$

In this equation

$$\mathscr{E}^*_{\alpha\beta} = 2e^4 N (v_n v_{n'})^2 \int \frac{k_\alpha k_\beta \delta(\mathbf{kv} - \mathbf{kv}')}{[\omega_k^2 - (\mathbf{kv})^2]^2} d^3k, \quad v_n = u_n/\gamma. \quad (22)$$

If one uses Eq. (9) of reference 1 instead of Eq. (1) for the random function $N_{\mathbf{qp}}(t)$, one can derive Eq. (21) directly.

The relativistic Maxwell distribution, which for particles of constant mass can be written in the form

$$f_1(q_i, p_i) = C\delta(p_i^2 + m_0^2c^2)\exp(p_i\bar{u}_i/\varkappa T), \quad (23)$$

satisfies the transport Eq. (17). In Eq. (23) C is a normalizing factor, $\bar{u}_i$ the average velocity fourvector in the equilibrium state, and T the temperature.

When one substitutes (23) into Eq. (17), one must take into account the fact that $k_i\bar{u}_i = 0$ because of the equation of continuity, and that $k_i u_i = k_i u_i' = 0$, because of the occurrence of the δ functions in the expression for the tensor $\mathscr{E}_{il}$.

In the system of coordinates in which the average electron velocity vanishes Eq. (23) for the usual distribution function $F^{(0)}(\mathbf{p})$ is of the form

$$F^{(0)}(\mathbf{p}) = C \exp\{-c\sqrt{m_0^2c^2 + p^2}/\varkappa T\}. \tag{24}$$

The transport equation (17) which we have obtained is incomplete even in the chosen approximation $\sim e^4$. This is due to the following facts. The derivation of Eq. (17) was based upon the use of Eq. (5), from which the variables characterizing the state of the electromagnetic field were completely eliminated. Indeed, in Eq. (5) we took only the retarded interaction between the particles into account.

We have seen that the transport equation obtained in this manner is not completely satisfactory, as it contains a divergent integral over the wave numbers, the "Coulomb" logarithm $\int dk/k$. A similar situation arose also when the corresponding nonrelativistic Landau transport equation was derived.[4]

The divergence difficulty was avoided in the nonrelativistic case by choosing an upper limit for the integral at large $k \sim 1/a$, where $a = e^2/m_0v^2$, and a lower limit at small $k \sim 1/r_d = k_d$, where r_d is the Debye radius. In a paper by Temko[5] the integration domain at small k in the derivation of the Fokker-Planck equation for a plasma is limited because the Debye screening is taken into account.

Both in this and in other cases, the interaction corresponding to small $k < 1/r_d$ is neglected in the derivation of the transport equation, which therefore takes only the screened interaction between the electrons in the plasma into account.

It is, however, well known[6,7] that the screened interaction accounts for only a part of the interaction energy of the electrons. The other part of the interaction, corresponding to small values of k, can be expressed in terms of the plasma oscillation variables. To obtain the complete transport equation one must thus take into account the interaction of non-equilibrium electrons with the plasma oscillations. One possible solution of this problem was considered by the author in reference 8.

We saw that a similar situation also arises in the derivation of a relativistic transport equation. This indicates that it is only possible to restrict oneself to taking retarded interactions between particles into account in the wave-number range $k > k_d$. It is thus impossible to eliminate the field variables completely. In order to choose the dividing value k_d, one must consider a suitable expression for the relativistic correlation function.

To conclude the present section, we note another possible way to derive a relativistic transport equation. One can use the fact that the interaction is weak to express approximately the second double-time distribution function which occurs in Eq. (7) in terms of the single-time distribution function. In that case one can reduce the problem not to the solution of Eqs. (14) and (15) for the double-time correlation functions, but to the solution of an equation for the single-time correlation function, which can also be obtained from Eq. (5).

2. A RELATIVISTIC FOKKER-PLANCK TRANSPORT EQUATION FOR A PLASMA WITH ACCOUNT OF THE EMISSION OF PLASMA WAVES

One can consider the distribution of the charged particles in a plasma to be in equilibrium when fast particles pass through a plasma, provided the number of fast (non-equilibrium) charged particles is small. The transport equation for the plasma is in that case the Fokker-Planck equation. One obtains it in the relativistic case from Eqs. (17) and (21) for the functions f_1 and F_1 by replacing in these equations the functions dependent on the primed momenta by the corresponding expressions for the equilibrium state. The relativistic Fokker-Planck transport equation for the function F_1 has thus, for instance, the following form

$$\frac{\partial F_1}{\partial t} + \mathbf{v}\frac{\partial F_1}{\partial \mathbf{q}} = \frac{\partial}{\partial p_\alpha} D_{\alpha\beta}\frac{\partial F_1}{\partial p_\beta} + \frac{\partial}{\partial p_\alpha}(A_\alpha F_1). \tag{25}$$

The coefficients $D_{\alpha\beta}$ and A_α are defined by the expressions

$$D_{\alpha\beta} = \int \mathscr{E}^*_{\alpha\beta}(\mathbf{p}, \mathbf{p}') F^{(0)}(\mathbf{p}')\, d^3p',$$

$$A_\alpha = -\int d^3p' \mathscr{E}^*_{\alpha\beta}(\mathbf{p}, \mathbf{p}')\, \partial F^{(0)}(\mathbf{p}')/\partial p'_\beta, \tag{26}$$

where $F^{(0)}(\mathbf{p})$ is the relativistic Maxwell distribution (24), while the tensor $\mathscr{E}^*_{\alpha\beta}$ is defined by Eq. (22).

We consider now briefly the derivation of the relativistic Fokker-Planck equation with account of diffusion processes and of the systematic friction processes caused by the excitation of plasma oscillations.

We turn to Eqs. (1) and (2). We expand the vector potential in a Fourier series

$$A_i = \sqrt{4\pi c^2/V}\sum_k \{A_i^{(1)}(k)\sin \mathbf{kq} + A_i^{(2)}(k)\cos \mathbf{kq}\}.$$

The equations for the Fourier coefficients have the form

$$\ddot{A}_i^{(j)} + \omega_k^2 A_i^{(j)} = \sqrt{4\pi/V}e\int N_{q_ip_i} \begin{matrix}\sin \mathbf{kq}\\ \cos \mathbf{kq}\end{matrix} d^3k d^4p, \quad j = 1,\ 2. \tag{27}$$

We eliminate, as before, the vector potential from the equations for $N_{q_ip_i}$, but we do this now only for the components of the vector potential with wave numbers $k > k_d$. The equation for the function $N_{q_ip_i}$ has in that case the form

$$u_i\partial N_{x_i}/\partial q_i + \frac{e}{c}F_{il}u_l\partial N_{x_i}/\partial p_i + \iint_{-\infty}^{t} L_{il}N_{x_i'}d\Omega' u_l\partial N_{x_i}/\partial p_i = 0, \quad x_i = (q_i,\ p_i). \tag{28}$$

In F_{il} we have here

$$A_i = \sqrt{4\pi c^2/V}\sum_{k<k_d} A_i^{(j)}(k) \begin{matrix}\sin \mathbf{kq}\\ \cos \mathbf{kq}\end{matrix}.$$

The equation for $A_i^{(j)}$ for $k < k_d$ is the same as Eq. (27). The tensor L_{il} in Eq. (28) is again defined by Eq. (4) with this difference, that the domain of integration is now restricted by the condition $k > k_d$.

We consider first the case where the plasma is in a state of statistical equilibrium. We shall show that we can in that case approximately separate the equations describing the behavior of particles with a screened interaction and the plasma oscillations. Such a kind of problem was considered by Bohm and Pines in their well-known papers[6,7] for a nonrelativistic plasma by a different method.

In an equilibrium state $\overline{N}_{x_i} = \overline{N}_{p_i}$ so that we can write $N_{x_i} = \overline{N}_{p_i} + \delta N_{x_i}$. Because we are dealing with a stationary and uniform case, the corresponding mean-square deviations depend only on coordinate and time differences, i.e., on $q_i - q_i'$.

We substitute the expression $N_{x_i} = \overline{N}_{p_i} + \delta N_{x_i}$ into Eq. (28). Taking into account the fact that in an equilibrium state $\overline{A}_i = 0$, $A_i = \delta A_i$, $F_{ik} = \delta F_{ik}$ we get for small deviations the following approximate equation for δN_{x_i}

$$u_i\partial\delta N_{x_i}/\partial q_i + \frac{e}{c}F_{il}u_l\partial\overline{N}_{p_i}/\partial p_i + \iint_{-\infty}^{t} L_{il}\delta N_{x_i'}d\Omega' u_l\partial\overline{N}_{p_i}/\partial p_i = 0. \tag{29}$$

Using Eq. (29) we eliminate δN_{x_i} from the right-hand side of Eq. (27) for $k < k_d$. The last term in Eq. (29) does not play any role here, as it contains only terms with $k > k_d$. The equation for the $A_i^{(j)}$ becomes

$$\ddot{A}_i^{(j)} + \Omega_k^2 A_i^{(j)} = 0. \tag{30}$$

The frequency Ω_k is different for different components of the vector $A_i^{(j)}$ and is determined from the corresponding dispersion relation for plasma oscillations.

The equation for the change δN_{x_i} due to the screened interaction between the particles has the following form

$$u_i\partial\delta N_{x_i}/\partial q_i + \iint_{-\infty}^{t} L_{il}\delta N_{x_i'}d\Omega' u_l\partial\overline{N}_{p_i}/\partial p_i = 0. \tag{31}$$

We can thus under the given assumptions separate the motion of the particles with screened interactions and the electromagnetic oscillations in the plasma.

We now consider non-equilibrium states in the system. The functions N_{x_i} will consist of two parts corresponding to equilibrium and non-equilibrium parts of the plasma $N_{x_i} = N_{x_i}^{(\text{non-eq})} + N_{x_i}^{(\text{eq})}$. We shall assume that the deviation from equilibrium is small ($N_{x_i}^{(\text{non-eq})} \ll N_{x_i}^{(\text{eq})}$) so that we can neglect any correlation between the non-equilibrium particles in the plasma.

The transition to equilibrium will in this case be determined by two factors, first the screened interaction between the non-equilibrium electrons and the equilibrium electrons in the plasma, and second the interaction with the plasma oscillations. The equation for $N_{x_i}^{(\text{eq})}$ which describes these processes has the following form

$$u_i\partial N_{x_i}^{(\text{non-eq})}/\partial q_i + \frac{e}{c}F_{il}u_l\partial N_{x_i}^{(\text{non-eq})}/\partial p_i + \iint_{-\infty}^{t} L_{il}u_l\{N_{x_i'}^{(\text{non-eq})}\partial N_{x_i}^{(\text{eq})}/\partial p_i + N_{x_i'}^{(\text{eq})}\partial N_{x_i}^{(\text{non-eq})}/\partial p_i\}\,d\Omega' = 0. \tag{32}$$

The electromagnetic field tensor F_{il} is determined in this equation by the value of the four-dimensional potential of the plasma waves excited by the non-equilibrium electrons

$$\ddot{A}_i^{(j)} + \Omega_k^2 A_i^{(j)} = \sqrt{\frac{4\pi}{V}}e\int u_iN_{x_i}^{(\text{non-eq})} \begin{matrix}\sin \mathbf{kq}\\ \cos \mathbf{kq}\end{matrix} d^3k d^4p. \tag{33}$$

Equations (32) and (33) constitute also a set from which we can obtain transport equations for the first distribution functions for the charged particles and the field oscillators. The method of obtaining these equations is similar to the one given in reference 8.

As in the foregoing, we restrict our considerations to the electrons in the plasma. The relativistic Fokker-Planck equation for the function F_1 differs then from Eq. (25) only through additional terms in the coefficients for the diffusion and for the systematic friction caused by the emission of plasma waves. We denote the corresponding extra terms by $D_{\alpha\beta}^{(em)}$ and $A_\alpha^{(em)}$. Each of them can be split into two parts determined respectively by the emission of transverse and of longitudinal waves.

The coefficients caused by the emission of transverse waves in an electron plasma differ from zero only if the plasma is in a decelerating medium. We do not consider this case here. The extra coefficients are thus completely determined by the emission of longitudinal plasma waves and are of the following form

$$D_{\alpha\beta}^{(\mathrm{em})} = \frac{e^2 \varkappa T}{2\pi} \int \frac{k_\alpha k_\beta}{k^2} \delta(\Omega_k^{\|} - \mathbf{kv})\, d^3k,$$

$$A_\alpha^{(\mathrm{em})} = \frac{e^2}{2\pi} \int \frac{k_\alpha \Omega_k^{\|}}{k^2} \delta(\Omega_k^{\|} - \mathbf{kv})\, d^3k. \qquad (34)$$

Equations (34) differ from the corresponding nonrelativistic expressions, obtained in reference 8, only in that now the plasma oscillation frequency is determined from the relativistic dispersion relation for longitudinal waves*

$$1 = \frac{4\pi e^2}{\varkappa T \omega} \int \frac{(u_x/\gamma)^2 F^{(0)}(p)}{\omega - u_x k/\gamma} d^3p, \quad k \| x, \quad \gamma = 1/\sqrt{1 - (v/c)^2}.$$

We consider now the limiting expressions for the coefficients for the diffusion and systematic friction. We find in the nonrelativistic approximation for the case of fast non-equilibrium particles ($v \gg \sqrt{\kappa T/m}$) from Eq. (26) the following expressions

$$A_x = A_y = 0, \quad A_z = \frac{e^2\omega_L^2}{v^2} \ln \frac{r_d}{a}, \quad v \| z, \quad a = \frac{e^2}{m_0 v^2};$$

$$D_{\alpha\beta} = 0 \quad \text{for } \alpha \neq \beta, \quad D_{zz} = \frac{e^2\omega_L^2}{v^3} \varkappa T \ln \frac{r_d}{a},$$

$$D_{xx} = D_{yy} = \frac{e^2\omega_L^2}{2v} m \ln \frac{r_d}{a}. \qquad (35)$$

The corresponding expressions for $D_{\alpha\beta}^{(em)}$ and $A_\alpha^{(em)}$ found from Eqs. (34) are of the form (compare with references 10 and 11)

$$A_x^{(\mathrm{em})} = A_y^{(\mathrm{em})} = 0, \quad A_z^{(\mathrm{em})} = \frac{e^2\omega_L^2}{v^2} \ln \frac{v}{r_d \omega_L} = \frac{e^2\omega_L^2}{v^2} \ln \frac{v}{v_T};$$

$$D_{\alpha\beta}^{(\mathrm{em})} = 0 \quad \text{for } \alpha \neq \beta, \quad D_{zz}^{(\mathrm{em})} = \frac{e^2\omega_L^2}{v^3} \varkappa T \ln \frac{v}{r_d \omega_L},$$

$$D_{xx}^{(\mathrm{em})} = D_{yy}^{(\mathrm{em})} = \frac{e^2\omega_L^2}{4v}. \qquad (36)$$

In the ultrarelativistic case, when the velocity of the non-equilibrium particles is close to that of light but the electron temperature is $\kappa T \ll m_0c^2$, the equations for the coefficients are the same as Eqs. (35) and (36), provided one replaces in them v by c. For the coefficients of the systematic friction we have, for instance,

$$A_z = \frac{e^2\omega_L^2}{c^2} \ln \frac{r_d}{a}, \quad A_z^{(\mathrm{em})} = \frac{e^2\omega_L^2}{c^2} \ln \frac{c}{r_d \omega_L}.$$

It is clear from the expressions given here that the total coefficients of the systematic friction and of diffusion along the direction of motion are independent of the choice of $k_d = 1/r_d$. In the ultrarelativistic approximation we get, for instance, the following expressions

$$A_z + A_z^{(\mathrm{em})} = \frac{e^2\omega_L^2}{c^2} \ln \frac{\delta}{a},$$

$$D_{zz} + D_{zz}^{(\mathrm{em})} = \frac{e^2\omega_L^2}{c^3} \varkappa T \ln \frac{\delta}{a}, \quad \delta = \frac{c}{\omega_L}.$$

One can easily obtain also the expressions for the coefficients for the case $\kappa T \gg m_0c^2$.

It follows from the formulae obtained here that taking the emission of plasma waves into account leads only to a slight change in the expressions under the logarithm sign. This consideration of the emission of plasma waves, done here, does therefore not lead to any essential change in the relaxation time. We must, finally, bear in mind that the results obtained here refer to that case where the number of non-equilibrium electrons is small and where the main plasma is in an equilibrium state.

We considered in reference 8, in connection with a discussion of Langmuir's paradox, an example where these conditions were not satisfied, i.e., where the initial state of the whole system was essentially non-equilibrium. It then turned out that, for instance, the transfer of energy of the electrons in the beam to the electrons in the plasma takes place over a time interval appreciably shorter than the relaxation time determined by the processes considered in the foregoing. The derivation of a transport equation for this case is a separate problem.

It is also of interest to use the chain of relativ-

*One must take u/γ instead of γu in the dispersion relations derived in Sec. 4 of reference 1.

istic equations for the distribution functions obtained in the present paper to obtain transport equations when there are external fields present and for a basis for the hydrodynamic approximation.

Some of the questions mentioned in the above will be considered in the future.

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