STABILITY OF EQUILIBRIUM OF A CONDUCTING LIQUID HEATED FROM BELOW IN A MAGNETIC FIELD

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The general effect of a uniform magnetic field on the stability of the equilibrium of a conducting liquid which is heated from below in a cavity of arbitrary shape is investigated. The time variation of the perturbations which arise in the liquid is always monotonic. The critical value of the Rayleigh number C_0^2, above which the equilibrium is unstable, increases monotonically with the Hartmann number M, so that the inequality in Eq. (4.16) is satisfied. At small values of M the critical value of the Rayleigh number is proportional to M^2 and the coefficient of proportionality can be computed. The asymptotic nature of the function $C_0(M)$ as $M \to \infty$ depends on the shape of the cavity and the direction of the field.

THE most interesting phenomena in magnetohydrodynamics occur when energy dissipation is not important. These effects have been investigated widely in recent years. Cases of motion in small volumes, in which case viscosity is important, are generally considered less interesting. Under these conditions, the magnetic field induces a current in the liquid and this current tends to retard the motion; in principle, no other additional effects arise. Cases of this kind have been considered by Hartmann[1] (plane Poiseuille flow), Chandrasekhar[2] (convection in a plane horizontal layer), Smirnov[3] (convection in a vertical tube heated from below), Velikhov[4] (Poiseuille flow and flow between rotating cylinders) and Regirer[5] (convection in a plane vertical slit).

Of the problems which have been considered, the convection problems are of special interest because of their connection with the theory of hydrodynamic stability. The initial state is an equilibrium state and the analysis of its stability is much simpler than the problem of flow stability. At the same time the presence of the magnetic field means that the equations of motion are not self-adjoint and hence more closely related to typical equations in the theory of hydrodynamic stability. If there is no magnetic field the problem is extremely simple.[6] Hence a general investigation of the effect of the magnetic field on the stability of equilibrium of a conducting liquid heated from below in an external uniform magnetic field is of interest.

1. EQUATIONS OF MOTION

We consider "slow" motion in a conducting liquid which is originally in equilibrium in a gravitational field

$$\mathbf{g} = -g\boldsymbol{\gamma}, \quad \boldsymbol{\gamma}^2 = 1 \tag{1.1}$$

and which is heated from below so that there is a constant vertical temperature gradient (as long as equilibrium prevails)

$$\nabla T_0 = -A\boldsymbol{\gamma}. \tag{1.2}$$

The liquid fills a cavity of arbitrary shape which is cut into an infinite external solid medium. The external magnetic field

$$\mathbf{H}_0 = H_0\boldsymbol{\beta}, \qquad \boldsymbol{\beta}^2 = 1 \tag{1.3}$$

produces a current density $\mathbf{j}$ and an additional magnetic field $\mathbf{h}$ in the liquid.

To the usual convection equations[7] it is now necessary to add the Lorentz force; since the motion is slow we retain only the linear perturbation terms. Then the equations of motion and the heat conduction equation are

$$\dot{\mathbf{v}} = -\rho^{-1}\nabla p - \nu \,\mathrm{curl\,curl}\,\mathbf{v} + \alpha g\boldsymbol{\gamma}T + (H_0/c\rho)[\mathbf{j}\times\boldsymbol{\beta}],$$
$$\dot{T} = A\boldsymbol{\gamma}\mathbf{v} + \chi\nabla^2 T, \qquad \mathrm{div}\,\mathbf{v} = 0. \tag{1.4}$$

In addition we have Maxwell's equations (in which the displacement current is neglected) and Ohm's law:

$$\mathrm{curl}\,\mathbf{h} = (4\pi\sigma/c)\{\mathbf{E} + c^{-1}H_0[\mathbf{v}\times\boldsymbol{\beta}]\} = 4\pi\mathbf{j}/c,$$
$$\dot{\mathbf{h}} = -c\,\mathrm{curl}\,\mathbf{E}, \quad \mathrm{div}\,\mathbf{h} = 0 \tag{1.5}$$

(σ is the electrical conductivity, $\mu = 1$, and $\mathbf{E}$ is the electric field).

We can eliminate $\mathbf{j}$ and $\mathbf{E}$ from these equations, expressing these quantities in terms of $\mathbf{h}$. Then the equations which describe the problem become

$$\dot{\mathbf{v}} = -\rho^{-1}\nabla p - \nu\,\mathrm{curl\,curl}\,\mathbf{v} + \alpha g \gamma T + (H_0/4\pi\rho)[\mathrm{curl}\,\mathbf{h}\times\boldsymbol{\beta}],$$
$$\dot{T} = \gamma A \mathbf{v} + \chi\nabla^2 T,$$
$$\dot{\mathbf{h}} = -(c^2/4\pi\sigma)\,\mathrm{curl\,curl}\,\mathbf{h} + H_0[\mathbf{v}\times\boldsymbol{\beta}],$$
$$\mathrm{div}\,\mathbf{v} = \mathrm{div}\,\mathbf{h} = 0. \tag{1.6}$$

For the external medium, in these equations we set the velocity equal to zero and replace the coefficients χ and σ by the corresponding coefficients in the medium, $\tilde{\chi}$ and $\tilde{\sigma}$:

$$\dot{T} = \tilde{\chi}\nabla^2 T, \quad \dot{\mathbf{h}} = -(c^2/4\pi\tilde{\sigma})\,\mathrm{curl\,curl}\,\mathbf{h}, \ \mathrm{div}\,\mathbf{h} = 0. \tag{1.6'}$$

At the boundaries of the cavity the velocity, magnetic field, temperature, and normal components of the current and heat flow must be continuous:

$$\mathbf{v}|_s = 0, \quad \mathbf{h}|_s = \tilde{\mathbf{h}}|_s, \quad T|_s = \tilde{T}|_s,$$
$$\mathbf{n}\,\mathrm{curl}\,\mathbf{h}|_s = \mathbf{n}\,\mathrm{curl}\,\tilde{\mathbf{h}}|_s, \quad \mathbf{n}\nabla T|_s = \eta\mathbf{n}\nabla\tilde{T}|_s \tag{1.7}$$

(η is the ratio of the heat conductivities of the medium and the liquid). Both T and **h** vanish at infinity.

We now introduce characteristic units: the characteristic dimension of the cavity l, H_0, and the characteristic velocity and temperature, defined by

$$v_1^2 = \left(\frac{H_0 c}{4\pi}\right)^2 \frac{1}{\rho\nu\sigma}, \quad T_1^2 = \left(\frac{H_0 c}{4\pi}\right)^2 \frac{A}{\alpha\chi\rho\sigma g}. \tag{1.8}$$

Then Eq. (1.6) assumes the following form:

$$\dot{\mathbf{v}} = -\nabla p - \mathrm{curl\,curl}\,\mathbf{v} + C\gamma T + M[\mathrm{curl}\,\mathbf{h}\times\boldsymbol{\beta}],$$
$$P\dot{T} = C\gamma\mathbf{v} + \nabla^2 T, \tag{1.9}$$
$$N\dot{\mathbf{h}} = -\mathrm{curl\,curl}\,\mathbf{h} + M\,\mathrm{curl}\,[\mathbf{v}\times\boldsymbol{\beta}],$$
$$\mathrm{div}\,\mathbf{v} = \mathrm{div}\,\mathbf{h} = 0;$$
$$\tilde{P}\dot{\tilde{T}} = \nabla^2\tilde{T}, \quad \tilde{N}\dot{\tilde{\mathbf{h}}} = -\mathrm{curl\,curl}\,\mathbf{h}, \ \mathrm{div}\,\mathbf{h} = 0. \tag{1.9'}$$

The boundary conditions (1.7) and the conditions at infinity remain the same as before.

We use the following dimensionless quantities in these equations:

$P = \nu/\chi = \tilde{\chi}\tilde{P}/\chi$ (Prandtl number)

$N = 4\pi\nu\sigma/c^2 = \sigma\tilde{N}/\tilde{\sigma}, \quad \eta = \tilde{\varkappa}/\varkappa,$

$C^2 = \alpha g A l^4/\nu\chi$ (Rayleigh number)

$M^2 = H_0^2\sigma l^2/\rho\nu c^2$ (Hartmann number squared)

(The so-called Lundquist number, which determines the nature of typical processes in magnetohydrodynamics, is $M\sqrt{N}$. This quantity does not contain the viscosity.)

The linear equations (1.9) do not contain the time explicitly so that all quantities may be assumed to be multiplied by a function of the form $e^{-\lambda t}$; we are then concerned with the boundary value problem:

$$\lambda\mathbf{v} = \nabla p + \mathrm{curl\,curl}\,\mathbf{v} - C\gamma T - M[\mathrm{curl}\,\mathbf{h}\times\boldsymbol{\beta}],$$
$$\lambda P T = -C\gamma\mathbf{v} - \nabla^2 T, \tag{1.10}$$
$$\lambda N\mathbf{h} = \mathrm{curl\,curl}\,\mathbf{h} - M\,\mathrm{curl}\,[\mathbf{v}\ \boldsymbol{\beta}],$$
$$\mathrm{div}\,\mathbf{v} = \mathrm{div}\,\mathbf{h} = 0;$$
$$\lambda\tilde{P}\tilde{T} = -\nabla^2 T, \quad \lambda\tilde{N}\tilde{\mathbf{h}} = \mathrm{curl\,curl}\,\mathbf{h}, \tag{1.10'}$$

with the boundary conditions (1.7) and the condition T = **h** = 0 at infinity. The sign of the real part of λ (the eigenvalue of the boundary-value problem) determines the stability: stability obtains when $\mathrm{Re}\,\lambda > 0$.

In what follows we will be concerned with integrals which are taken over all space. These are written in the form of sums of integrals over the volume of the liquid and the external volume whose integrands differ from each other by the obvious substitution of P by $\tilde{P}$ and so on. The boundary conditions allow us to use Gauss' theorem over all space since the integrals over the surface of the interface always cancel.

2. STABILITY IN THE ABSENCE OF A MAGNETIC FIELD

This problem has been investigated by one of us.[6] In the absence of a field M = 0 and Eq. (1.10) can be simplified:

$$\lambda\mathbf{v} = \nabla p + \mathrm{curl\,curl}\,\mathbf{v} - C\gamma T,$$
$$P\lambda T = -C\gamma\mathbf{v} - \nabla^2 T, \ \mathrm{div}\,\mathbf{v} = 0; \tag{2.1}$$
$$\tilde{P}\lambda\tilde{T} = -\nabla^2\tilde{T}. \tag{2.1'}$$

The system of equations in (2.1) and (2.1') is self-adjoint and the eigenvalues λ and eigenfunctions **v** and T are real. The equations in (2.1) are the Euler equations for the variational problem

$$J[\mathbf{v}, T] = \frac{1}{2}\int\{(\mathrm{curl}\,\mathbf{v})^2 + (\nabla T)^2 - 2C\gamma\mathbf{v}T\}\,dV + \frac{1}{2}\eta\int(\nabla\tilde{T})^2\,d\tilde{V} = \mathrm{extr},$$
$$K[\mathbf{v}, T] = \frac{1}{2}\int\{\mathbf{v}^2 + PT^2\}\,dV + \frac{1}{2}\eta\int\tilde{P}\tilde{T}^2\,d\tilde{V} = 1,$$
$$\mathrm{div}\,\mathbf{v} = 0 \tag{2.2}$$

with the earlier conditions (1.7), while the λ's are stationary values of the quotient:

$$\lambda = \mathrm{extr}\,(J/K). \tag{2.3}$$

The last situation, which does not appear in problems of hydrodynamic stability, makes the analysis much easier. For a given C there exists an infinite set of solutions

$$\lambda_n;\ \{\mathbf{v}_n, T_n, p_n\},\ n = 0, 1, 2, \ldots, \tag{2.4}$$

which can be designated in order of increasing λ.

These solutions are orthogonal to each other in the following sense:

$$\frac{1}{2}\int\{\mathbf{v}_m\mathbf{v}_n + PT_mT_n\}\,dV + \frac{1}{2}\eta\int\widetilde{P}\,\widetilde{T}_m\widetilde{T}_n d\widetilde{V} = \delta_{mn},$$

$$\frac{1}{2}\int\{\mathrm{curl}\,\mathbf{v}_m\mathrm{curl}\,\mathbf{v} + \nabla T_m\cdot\nabla T_n - C\gamma\mathbf{v}_m T - C\gamma\mathbf{v}_n T_m\}\,dV$$

$$+\frac{1}{2}\eta\int\nabla\widetilde{T}\cdot\nabla\widetilde{T}_n\,d\widetilde{V} = \lambda_n\delta_{mn}. \qquad (2.5)$$

It is apparent that $J > 0$ when $C = 0$, that is to say, all the $\lambda_n > 0$, and the liquid is stable. By means of a variational technique (2.3) it is shown in reference 6 that when C is increased all the λ_n are diminished, i.e., the damping of the perturbations becomes weaker and for some value $C = C_0$ the eigenvalue λ_0 vanishes. When $C > C_0$, the zeroth motion increases in time and the liquid becomes unstable. With a further increase in C we obtain successive negative values λ_1, λ_2, etc.

In order to find the critical number C_α we solve Eq. (2.1), setting $\lambda = 0$:

$$C\gamma T = \nabla p + \mathrm{curl}\,\mathrm{curl}\,\mathbf{v},\quad C\gamma\mathbf{v} = -\nabla^2 T,\qquad \mathrm{div}\,\mathbf{v} = 0; \qquad (2.6)$$

$$\nabla^2\widetilde{T} = 0 \qquad (2.6')$$

with the earlier boundary conditions. Equations (2.6) and (2.6′) are also self-adjoint and are equivalent to the variational problem:

$$I[\mathbf{v}, T] = \frac{1}{2}\int\{(\mathrm{curl}\,\mathbf{v})^2 + (\nabla T)^2\}\,dV$$

$$+\frac{1}{2}\eta\int(\nabla\widetilde{T})^2 d\widetilde{V} = \mathrm{extr},$$

$$Q[\mathbf{v}, T] = \int\gamma\mathbf{v}T\,dV = 1,\ \mathrm{div}\,\mathbf{v} = 0. \qquad (2.7)$$

The eigenvalues for this problem are given by

$$C = \mathrm{extr}\,(I/Q). \qquad (2.8)$$

We shall call the solutions of this problem the critical motions, enumerating them by Greek subscripts in order of increasing C:

$$C_\alpha;\ \{\mathbf{v}_\alpha, T_\alpha, p_\alpha\}, \qquad (2.9)$$

so that C_α^2 is the value of the Rayleigh number at which perturbations characterized by $n < \alpha$ increase, the perturbation characterized by $n = \alpha$ is neutral, and perturbations characterized by $n > \alpha$ are damped. The critical motions satisfy the following orthogonality conditions:

$$\int\gamma\mathbf{v}_\alpha T_\beta dV = \delta_{\alpha\beta},$$

$$\int\mathrm{curl}\,\mathbf{v}_\alpha\,\mathrm{curl}\,\mathbf{v}_\beta dV = C_\alpha\delta_{\alpha\beta},$$

$$\int\nabla T_\alpha\cdot\nabla T_\beta dV + \eta\int\nabla\widetilde{T}_\alpha\cdot\nabla\widetilde{T}_\beta\,d\widetilde{V} = C_\alpha\delta_{\alpha\beta}. \qquad (2.10)$$

It is assumed that Eq. (2.9) defines a complete set and that this set can be used for expansion of any **v**, T, and p.

3. PERTURBATIONS IN A MAGNETIC FIELD

The system of equations in (1.10) for perturbations in a magnetic field is not self-adjoint and it is not immediately apparent whether or not the eigenvalues are real. From Eq. (1.10) it is easy to derive the integral relation

$$(\lambda - \lambda^*)\left\{\int[\mathbf{v}^*\mathbf{v} + PT^*T - N\mathbf{h}^*\mathbf{h}]\,dV\right.$$

$$\left.+\int[\eta\widetilde{P}\,T^*T - \widetilde{N}\,\widetilde{\mathbf{h}}^*\widetilde{\mathbf{h}}]\,d\widetilde{V}\right\} = 0, \qquad (3.1)$$

which, strictly speaking, does not prove that the λ are real, although it does suggest that such is the case. It is possible, however, to expand the eigenvalues and eigenfunctions in powers of M^2 and these expansions are real.

These expansions are obtained as follows. When $M = 0$ Eq. (10) has a set of solutions given by (2.4). Physically it is clear that this set must be complete in the sense that any pair of functions $\{\mathbf{v}, T\}$ can be expanded in functions of the set:

$$\{\mathbf{v}, T\} = \sum_n a_n\{\mathbf{v}_n, T_n\} \qquad (3.2)$$

with the same coefficients a_n in both expansions.

We may note that a set of several $\mathbf{v}_n$ (or T_n) is overcomplete for any **v**; hence any **v** can be expanded in different ways in the $\mathbf{v}_n$, depending on the T to which this **v** is related. The reason such expansions are real is clear from the fact that any initial (**v**, T) vary in accordance with Eq. (1.9); i.e., the expansions must consist of terms which vary exponentially in time.

In Eq. (1.10) we now write

$$\lambda = \lambda_n + M^2\lambda^{(1)} + \ldots + M^{2k}\lambda^{(k)} + \ldots, \qquad (3.3)$$

$$\{\mathbf{v}, T, p\} = \{\mathbf{v}_n, T_n, p_n\}$$

$$+\sum_{m\neq n}[M^2 a_m^{(1)} + \ldots + M^{2k}a_m^{(k)} + \ldots]\{\mathbf{v}_m, T_m, p_m\}, \qquad (3.4)$$

$$\mathbf{h} = M\mathbf{h}^{(1)} + \ldots + M^{2k-1}\mathbf{h}^{(k)} + \ldots\,. \qquad (3.5)$$

Assuming that all terms of these expansions are determined to order $(k-1)$, we now show that it is possible to determine terms of order k. If we substitute the expansions (3.3) and (3.4) in the third equation of (1.10) and consider M^{2k-1} terms we have

$$N\lambda_n\mathbf{h}^{(1)} - \mathrm{curl}\,\mathrm{curl}\,\mathbf{h}^{(1)} = -\,\mathrm{curl}\,[\mathbf{v}_n\times\boldsymbol{\beta}] \quad (k = 1);$$

$$N\lambda_n\mathbf{h}^{(k)} - \mathrm{curl}\,\mathrm{curl}\,\mathbf{h}^{(k)} = -\,\mathrm{curl}\sum_{m\neq n}a_m^{(k-1)}[\mathbf{v}_m\times\boldsymbol{\beta}]$$

$$-[\lambda^{(1)}\mathbf{h}^{(k-1)} + \ldots + \lambda^{(k-1)}\mathbf{h}^{(1)}] \qquad (k > 1). \quad (3.6)$$

The right sides of these equations are assumed to be known and, solving Eq. (3.6) with the imposed boundary conditions, we obtain the $\mathbf{h}^{(k)}$. We then substitute the expansions in (3.3) and (3.4) and the

$\mathbf{h}^{(k)}$, which are now known, in the remaining equations of (1.10) and consider M^{2k} terms. In this way we obtain

$$\lambda^k \mathbf{v}_n + \sum_{m \neq n} [\lambda^{(k-1)} a_m^{(1)} + \ldots + \lambda_n a_m^{(k)}] \mathbf{v}_m = \sum_{m \neq n} a_m^{(k)} [\nabla p_m + \operatorname{curl} \operatorname{curl} \mathbf{v}_m - C\gamma T_m] - [\operatorname{curl} \mathbf{h}^{(k)} \times \beta],$$

$$P\left\{ \lambda^{(k)} T_n + \sum_{m \neq n} [\lambda^{(k-1)} a_m^{(\,)} + \ldots + \lambda_n a_m^{(k)}]\, T_m \right\} = -\sum_{m \neq n} a_m^{(k)} [C\gamma \mathbf{v}_m + \nabla^2 T_m] \tag{3.7}$$

and similar equations in the external medium.

Multiplying Eq. (3.7) by

$$\{\mathbf{v}_n, T_n\}$$

and integrating, by virtue of the orthogonality condition (2.5) we obtain

$$\lambda^{(k)} = \int \operatorname{curl} \mathbf{h}^{(k)} [\mathbf{v}_n \times \beta]\, dV, \tag{3.8}$$

then, multiplying by

$$\{\mathbf{v}_m, T_m\} \quad (m \neq n)$$

and integrating we obtain

$$[\lambda^{(k-1)} a_m^{(1)} + \ldots + \lambda_n a_m^{(k)}] = \lambda_m a_m^{(k)} + \int \operatorname{curl} \mathbf{h}^{(k)} [\mathbf{v}_m \times \beta]\, dV,$$

whence

$$a_m^{(k)} = \frac{1}{\lambda_m - \lambda_n} \left\{ \lambda^{(k-1)} a_m^{(1)} + \ldots + \lambda^{(1)} a_m^{(k-1)} - \int \operatorname{curl} \mathbf{h}^{(k)} [\mathbf{v}_m \times \beta]\, dV \right\}. \tag{3.9}$$

By this method it is then possible to compute successively all terms of the expansions in Eqs. (3.3) to (3.7); it is obvious that when $M \neq 0$ the solutions are real. We may note that the following relation follows from Eqs. (3.3) and (3.8):

$$\lambda = \lambda_n + M \int \operatorname{curl} \mathbf{h} [\mathbf{v}_n \times \beta]\, dV, \tag{3.10}$$

which can also be derived directly from Eq. (1.10).

Taking account of the fact that $\{\lambda, \mathbf{v}, T\}$ are real, from Eq. (1.10) we find that

$$\lambda = \left\{ \int \Big[(\operatorname{curl} \mathbf{v})^2 + (\nabla T)^2 - (\operatorname{curl} \mathbf{h})^2 - 2C\gamma \mathbf{v} T - 2M \operatorname{curl} \mathbf{h} [\beta \times \mathbf{v}] \Big] dV + \int [\eta (\nabla \widetilde{T})^2 - (\operatorname{curl} \mathbf{h})^2]\, dV \right\} \left\{ \int [\mathbf{v}^2 + PT^2 - N \mathbf{h}^2]\, dV + \int [\eta \widetilde{P} \widetilde{T}^2 - \widetilde{N}^2 \mathbf{h}^2]\, dV \right\}^{-1}, \tag{3.11}$$

and Eq. (1.10) is obtained by the variation of the numerator of this expression with the denominator held constant.

We cannot show starting directly from Eq. (3.11) that in a magnetic field there will be a critical Rayleigh number C_0^2 at which the motion becomes unstable, although this result is almost obvious physically. But if stability is lost at the critical value C_0, the smallest λ is zero. Hence it is possible to investigate Eq. (1.10) directly with $\lambda = 0$, assuming that M is given. Thus we can obtain the spectrum of critical $C_\alpha(M)$.

4. STABILITY IN A MAGNETIC FIELD

Writing $\lambda = 0$ in Eq. (1.10) we obtain the equations for the critical motions. In these equations it is convenient to introduce the current

$$\mathbf{j} = \operatorname{curl} \mathbf{h}, \tag{4.1}$$

so that the equations assume the form

$$C\gamma T = \nabla p + \operatorname{curl} \operatorname{curl} \mathbf{v} - M [\mathbf{j} \times \beta], \quad C\gamma \mathbf{v} = -\nabla^2 T,$$
$$\operatorname{curl} \{\mathbf{j} - M [\mathbf{v} \times \beta]\} = 0, \quad \operatorname{div} \mathbf{v} = \operatorname{div} \mathbf{j} = 0. \tag{4.2}$$

In the external medium

$$\nabla^2 \widetilde{T} = 0, \quad \operatorname{curl} \mathbf{j} = 0, \quad \operatorname{div} \mathbf{j} = 0. \tag{4.2'}$$

The boundary conditions remain the same as before except that we must take account of the fact that the normal component of the current is continuous.

It is easy to show that Eq. (4.2) is obtained by solution of the following variational problem:

$$I[\mathbf{v}, T, \mathbf{j}] = \frac{1}{2} \int \{(\operatorname{curl} \mathbf{v})^2 + (\nabla T)^2 - \mathbf{j}^2 - 2M \mathbf{j} [\beta \times \mathbf{v}]\}\, dV + \frac{1}{2} \int \{\eta (\nabla \widetilde{T})^2 - \widetilde{\mathbf{j}}^2\}\, d\widetilde{V} = \text{extr};$$
$$Q[\mathbf{v}, T] = \int \gamma \mathbf{v} T\, dV = 1, \quad \operatorname{div} \mathbf{v} = \operatorname{div} \mathbf{j} = 0 \tag{4.3}$$

with the same boundary conditions as before. The quantity $C(M)$ can itself be expressed in terms of the extreme of the function $\mathbf{v}$, T, $\mathbf{j}$ by use of any of the following formulas:

$$C(M) = I[\mathbf{v}, T, \mathbf{j}] / Q[\mathbf{v}, T] = \left\{ \int (\nabla T)^2 dV + \eta \int (\nabla \widetilde{T})^2 d\widetilde{V} \right\} \Big/ \gamma \mathbf{v} T\, dV = \int [(\operatorname{curl} \mathbf{v})^2 + \mathbf{j}^2]\, dV \Big/ \int \gamma \mathbf{v} T\, dV, \tag{4.4}$$

where the following relations obtain for the critical values of $\mathbf{v}$, T and $\mathbf{j}$

$$\int \mathbf{j}^2 dV = M \int \mathbf{j} [\mathbf{v} \times \beta]\, dV,$$
$$\int [(\operatorname{curl} \mathbf{v})^2 + \mathbf{j}^2]\, dV = \int (\nabla T)^2 dV + \eta \int (\nabla \widetilde{T})^2 d\widetilde{V}. \tag{4.5}$$

We now compute the derivative of $C(M)$ with respect to M from the first relation in Eq. (4.4). It is apparent that it is necessary to differentiate only with respect to M, which appears explicitly in the integral I, since the result of differentiation of the functions $\mathbf{v}$, T and $\mathbf{j}$ gives zero by virtue of Eq. (4.2). Consequently

$$dC(M)/dM = \int \mathbf{j}[\mathbf{v}\times\boldsymbol{\beta}]\,dV \Big/ \int \gamma \mathbf{v} T\,dV \qquad (4.6)$$

or, from Eqs. (4.5) and (4.6),

$$\frac{M}{C}\frac{dC}{dM} = \int \mathbf{j}^2 dV \Big/ \left\{\int (\nabla T)^2 dV + \eta \int (\nabla \widetilde{T})^2 d\widetilde{V}\right\}. \quad (4.7)$$

It follows from Eq. (4.5) that

$$0 \leqslant \int \mathbf{j}^2 dV \Big/ \left\{\int (\nabla T)^2 dV + \eta \int (\nabla \widetilde{T})^2 d\widetilde{V}\right\}$$
$$= 1 - \int (\operatorname{curl}\mathbf{v})^2 dV \Big/ \left\{\int (\nabla T)^2 dV + \eta \int (\nabla \widetilde{T})^2 d\widetilde{V}\right\} < 1, \qquad (4.8)$$

whence

$$0 \leqslant d\ln C / d\ln M < 1. \qquad (4.9)$$

At small values of M the dependence of C (M) on M can be determined easily by a perturbation method.* We consider the change, under the effect of a weak magnetic field, in the critical motion ($\mathbf{v}_0$, T_0, p_0, $h_0 = 0$) which corresponds to the smallest critical Rayleigh number $C_0^2(0)$. We expand all quantities in terms of the critical motion (without the magnetic field) (2.9):

$$\mathbf{v} = \mathbf{v}_0 + M^2 \sum_{\alpha\neq 0} \beta_\alpha \mathbf{v}_\alpha \ldots,$$
$$T = T_0 + M^2 \sum_\alpha \theta_\alpha T_\alpha + \ldots,$$
$$p = p_0 + M^2 \sum_\alpha \pi_\alpha p_\alpha + \ldots,$$
$$C_0(M) = C_0(0) + \Delta C. \qquad (4.10)$$

(In the summation for $\mathbf{v}$ we neglect the term with $\alpha = 0$, which is equivalent to a change of normalization.) Substituting in Eq. (4.2) and keeping only the lowest power of M we have

$$M^2\Big[-\sum_\alpha \nabla \pi_\alpha p_\alpha - \sum_{\alpha\neq 0} \beta_\alpha \operatorname{curl}\operatorname{curl}\mathbf{v}_\alpha + C_0(0)\sum_\alpha \theta_\alpha \gamma T_\alpha\Big]$$
$$+ \Delta C \gamma T_0 + M[\mathbf{j}\times\boldsymbol{\beta}] = 0,$$
$$M^2 C_0(0)\gamma \sum_{\alpha\neq 0} \beta_\alpha \mathbf{v}_\alpha + \Delta C \gamma \mathbf{v}_0 + M^2 \sum_\alpha \theta_\alpha \nabla^2 T_\alpha = 0,$$
$$\mathbf{j} = M[\mathbf{v}_0\times\boldsymbol{\beta}] - \nabla\varphi. \qquad (4.11)$$

The current may be assumed known since in determining the potential φ from $\operatorname{div}\mathbf{j} = 0$, we get

$$\nabla^2\varphi = M \operatorname{div}[\mathbf{v}_0\times\boldsymbol{\beta}]. \qquad (4.12)$$

It is apparent that the current is proportional to M.

In order to compute ΔC we multiply Eq. (4.11) by $\mathbf{v}_0$ and T_0 respectively, integrate over all space, and add. By virtue of the orthogonality condition (2.10) we have

$$\Delta C + M^2 C_0(0)\theta_0 + M\int [\mathbf{j}\times\boldsymbol{\beta}]\mathbf{v}_0 dV = 0,$$
$$\Delta C - M^2 C_0(0)\theta_0 = 0,$$

whence

$$\Delta C = \frac{1}{2} M \int \mathbf{j}[\mathbf{v}_0\times\boldsymbol{\beta}]\,dV. \qquad (4.13)$$

It follows from the last equation of (4.11) that

$$\frac{1}{2}\int \mathbf{j}[\mathbf{j}+\nabla\varphi]\,dV = \frac{1}{2}\int \mathbf{j}^2 dV - \frac{1}{2}\int \varphi \operatorname{div}\mathbf{j}\,dV,$$

i.e.,

$$\Delta C = \frac{1}{2}\int \mathbf{j}^2 dV. \qquad (4.14)$$

The current in this expression is computed from the velocity $\mathbf{v}_0$, normalized in accordance with Eq. (2.10). Thus, $\Delta C \sim M^2$.

If we return to the conventional units, in accordance with Eqs. (1.8) and (2.10)

$$\Delta C / C_0(0) = \frac{1}{2}\int (\mathbf{j}^2/\sigma)\,dV \Big/ \int \nu\rho(\operatorname{curl}\mathbf{v})^2 dV, \qquad (4.15)$$

so that the change in the critical Rayleigh number in the magnetic field is determined by the ratio of the Joule heat to the viscous dissipation.

The function $C_0(M)$, which is positive for M = 0, first increases as M^2 and then continues to increase monotonically, so that

$$dC_0(M)/dM < C_0(M)/M. \qquad (4.16)$$

The inequality given here cannot become an equality as this would mean that the curl of the velocity vanishes everywhere, (zero velocity everywhere), which cannot be the case. Consequently, the curve $C_0(M)$ at each point intersects a line drawn to this point from the origin of coordinates and cannot touch this line at any other point. When $M \to \infty$ the quantity C_0 cannot increase faster than M but may also approach a constant. This follows from the simple example of convection between two parallel vertical planes.[5] When the external magnetic field is vertical it has no effect on the zeroth critical perturbation, for which the velocity is also vertical, and the critical value C_0 is independent of M. In the case of a horizontal field, however,

$$C_0(M) = \pi^2[1 + M^2/\pi^2]^{1/2},$$
$$C_0(M) \sim \pi M \text{ for } M \to \infty.$$

*This calculation has been carried out by S. V. Ust'-Kachkintseva at the Perm University.

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