ON THE NONLINEAR THEORY OF STATIONARY PROCESSES IN AN ELECTRON PLASMA

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We have used a transport equation without a collision integral to consider nonlinear problems of oscillation modes of an electron plasma and of the interaction between plasma beams. We have obtained relations between the wavelength and the frequency and the amplitude of the oscillations. We determine the maximum field for which periodic processes are possible in a plasma. We have found the conditions for the propagation of large amplitude waves both for a plasma at rest and for a plasma of moving beams.

THE Boltzmann-Vlasov transport equation,[1] which is based upon a model of particles that interact through a self-consistent field, is a good approximation to describe an electron plasma. The field describes the so-called long-range collisions of the particles, i.e., the interaction of the particles which is connected with the slow decrease of the Coulomb forces. The short-range interaction is described by the collision integral.

This paper is devoted to a study of stationary processes in an electron plasma using this equation. We consider in the paper the propagation of electrostatic waves with a constant phase velocity both in a plasma at rest and in a plasma of moving beams in the nonlinear approximation. Similar nonlinear problems have been solved earlier by a number of authors;[2-4] they considered the interaction of beams in the hydrodynamic approximation[2] and the interaction of a hydrodynamic beam with a plasma described by a transport equation.[3,4]

For the sake of simplicity, we consider in this paper an unbounded plasma. We assume that all quantities depend on one coordinate x only. The ion motion and effects of short-range collisions both with charged and with neutral particles are not taken into account. The first neglect is based upon the large difference in mass between ions and electrons, and the second one can be assumed to be fulfilled if we consider a high temperature plasma with a high degree of ionization.

To consider solutions of the type of moving waves we shall go over to a system of coordinates in which the wave is at rest. Under the above assumptions the electron plasma is described in this system of coordinates by the equations

$$\frac{\partial f}{\partial x}+\frac{e}{m}\frac{d\varphi}{dx}\frac{\partial f}{\partial v}=0, \qquad \frac{d^2\varphi}{dx^2}=4\pi e\left\{\int\limits_{(v)} f(x, v)\,dv-n_0\right\}, \quad (1)$$

where $f(x, v)$ is the electron distribution function in configuration space, $\varphi(x)$ the self-consistent field potential, e and m the electronic charge and mass, and n_0 the constant positive charge density. We must say a few words about the domain of integration over the velocities in the Laplace equation. One usually integrates between the limits $-\infty < v < +\infty$, i.e., one assumes that at every point in space there are particles with zero kinetic energy. We shall consider an electron plasma in which there are no bound particles, i.e., where there are no particles for which the region of motion is limited on two or on one side by the electrostatic potential barrier. The limits of integration are in that case determined as follows.

Let $\mathscr{E}_0$ be the total energy of that plasma particle at $x = x_0$ which possesses a minimum energy. The other particles at that point will then satisfy the relation

$$mv^2/2-e\varphi(x_0)\geqslant\mathscr{E}_0.$$

Since the particle with minimum energy performs an infinite motion during which its energy is conserved, the relation

$$mv^2/2-e\varphi(x)\geqslant\mathscr{E}_0$$

must be valid for all particles at any point x. This relation determines the domain of integration over the velocities.

A particle may be captured if it loses energy through an interaction that is not connected with the self-consistent field. The capture of a particle leads to a decrease of the depth of the potential well of the self-consistent field. This is the normal mechanism of damping of the field when short-range Coulomb collisions and collisions with neutral atoms are taken into account. These processes are not taken into account in the present paper, since we have assumed that the interaction takes place only through the self-consistent field $E(x)$.

The presence of captured particles is connected with the character of the establishment of a stationary state in the plasma. In the present paper we assume that it occurs without capture of particles. For waves whose phase velocity is much larger than the average thermal velocity of the electrons, this assumption is well justified.

Let us find the electrical field distribution in the plasma in its dependence on the form of the distribution function and the magnitude of this field in an arbitrary point $x = 0$; i.e., let us solve the Cauchy problem for Eq. (1) under the following initial conditions:

$$\varphi(0) = 0, \quad -d\varphi(0)/dx = E_0,$$
$$f(0, v) = A\exp\{-m(v - v_{\mathrm{ph}})^2/2T\} \quad \text{if} \quad mv^2/2 \geqslant mv_0^2/2,$$
$$f(0, v) = 0 \quad \text{if} \quad mv^2/2 < mv_0^2/2, \tag{2}$$

where $\frac{1}{2}mv_0^2$ is the minimum energy at the point $x = 0$, v_{ph} the constant phase velocity of the wave, T the electron temperature in ergs, and A is determined from the normalization condition

$$A\int\limits_{v^2\geqslant v_0^2} \exp\{-m(v - v_{\mathrm{ph}})^2/2T\}\,dv = n_0. \tag{3}$$

The form of $f(0, v)$ and the normalization condition are chosen in such a way that the electron velocity distribution in the laboratory system of coordinates goes over into the Maxwell distribution for $E(x) \to 0$.

The quantities v_0^2 and E_0 are connected with the character of the establishment of the self-consistent field. One may assume for the oscillations of a free plasma that the field E_0 is caused by the density fluctuations of the electrons. In the following we shall assume that $v_0^2 \neq 0$. Analysis shows that if $v_0^2 = 0$ a perturbation connected with the density fluctuations of the electrons will be concentrated in a bounded region. Its dimensions are determined by the perturbing field. The detailed problem of the choice of v_0^2 is considered later on.

The distribution function satisfying the transport equation and the initial conditions is as follows

$$f(x, v) = \begin{cases} A\exp\left[-\dfrac{m}{2T}\left(\pm\sqrt{v^2 - \dfrac{2e\varphi}{m}} - v_{\mathrm{ph}}\right)^2\right] \\ \quad \text{if} \quad \dfrac{mv^2}{2} - e\varphi \geqslant \dfrac{mv_0^2}{2} \\ 0 \qquad \text{if} \quad mv^2/2 - e\varphi < mv_0^2/2 \end{cases} \tag{4}$$

The $\pm$ signs in front of the radical correspond to $v > 0$ and $v < 0$.

It is convenient for the following to introduce the dimensionless quantities

$$mv^2/2T = u^2, \; e\varphi/T = z, \; x\sqrt{4\pi e^2 n_0/T} = x/\lambda_D = \zeta,$$
$$mv_0^2/2T = u_0^2; \; mv_{\mathrm{ph}}^2/2T = u_{\mathrm{ph}}^2. \tag{5}$$

The Laplace equation takes on the following form in this notation

$$\frac{d^2z}{d\zeta^2} = \frac{\int\limits_{u_0}^{\infty}\{\exp[-(t-u_{\mathrm{ph}})^2] + \exp[-(t+u_{\mathrm{ph}})^2]\}(t^2+z)^{-1/2}\,t\,dt}{\int\limits_{u_0}^{\infty}\{\exp[-(t-u_{\mathrm{ph}})^2] + \exp[-(t+u_{\mathrm{ph}})^2]\}\,dt} - 1. \tag{6}$$

Here $u_0 = +\sqrt{u_0^2}$. Integrating this equation twice and taking the initial conditions into account gives

$$\zeta = \int\limits_0^z dz/\sqrt{\widetilde{E}_0^2 - 2V(z)}, \tag{7}$$

where

$$\widetilde{E}_0^2 = (dz(0)/d\zeta)^2 = E_0^2/4\pi n_0 T,$$

$$V(z) = z - 2\,\frac{\int\limits_{u_0}^{\infty}\{\exp[-(t-u_{\mathrm{ph}})^2] + \exp[-(t+u_{\mathrm{ph}})^2]\}(\sqrt{t^2+z} - t)\,t\,dt}{\int\limits_{u_0}^{\infty}\{\exp[-(t-u_{\mathrm{ph}})^2] + \exp[-(t+u_{\mathrm{ph}})^2]\}\,dt}, \tag{8}$$

$V(z)$ is defined in the interval $-u_0^2 \le z < \infty$. $V(z)$ decreases monotonically in the section $-u_0^2 \le z \le 0$, and increases monotonically in the section $0 \le z < \infty$. The point $z = 0$ is a minimum of $V(z)$. It is well known (see, for instance, reference 5) that (7) defines z as a periodic function of ζ for those z which satisfy the condition

$$2V(z) \leqslant \widetilde{E}_0^2. \tag{9}$$

This relation, however, defines for $Z = -u_0^2$ the maximum field strength $\widetilde{E}_{0,\max}$ for which stationary processes are possible in a plasma which is described by Eq. (1).

The period $\widetilde{\lambda}$ of the function z is determined by the usual expression

$$\widetilde{\lambda} = 2\int\limits_{z_1}^{z_2}\frac{dz}{\sqrt{\widetilde{E}_0^2 - 2V(z)}}, \tag{10}$$

where z_1 and z_2 are the roots of the equation

$$\widetilde{E}_0^2 - 2V(z) = 0. \tag{11}$$

The difference $z_2 - z_1$ determines the amplitude of the self-consistent field potential in its dependence on the magnitude of $\widetilde{E}_0$.

Equations (7) — (11) solve the given problem about the propagation of electrostatic waves of constant phase velocity in an electron plasma. All required results can in the general case be obtained by numerical integration of the corresponding equations.

We need an explicit expression for $\widetilde{E}$ and u_0 for further evaluations. One can see from (8) that $\widetilde{E}_0$ is determined by the ratio of the energy density of the electrostatic field to the average kinetic energy density of the electrons. For fields which are caused by fluctuation processes this quantity is of the order of unity. We shall choose u_0^2 from thermodynamic considerations, assuming that an electron plasma in its equilibrium state is a perfect gas described by the distribution function

$$f(u)\,du = \pi^{-1/2}\exp\{-(u-u_{ph})^2\}\,du. \quad (12)$$

The factor $\pi^{-1/2}$ corresponds to a normalization to one particle. The most probable value of u_0^2 will be the average value $\overline{u^2}$. One can then write

$$u_0^2 = \overline{u}^2 - \Delta u^2, \quad (13)$$

where Δu^2 is of the same order of magnitude as the average fluctuations in energy $\sqrt{\overline{(\Delta u^2)^2}}$. From the physics of the problem, u_0^2 corresponds to the minimum energy so that we must take for Δu^2 a quantity somewhat larger than $\sqrt{\overline{(\Delta u^2)^2}}$, or,

$$\Delta u^2 = A\sqrt{\overline{(\Delta u^2)^2}},\ A \gtrsim 1.$$

From these considerations, however, it follows that we must choose the minus sign in front of Δu^2. From (12) we get

$$\overline{u}^2 = {}^1/_2 + u_{ph}^2,\quad \sqrt{\overline{(\Delta u^2)^2}} = \sqrt{\overline{u^4} - (\overline{u^2})^2} = \sqrt{2u_{ph}^2 + {}^1/_2}. \quad (14)$$

We see from (13) and (14) that if $u_{ph}^2 \lesssim 1$ the magnitude of u_0^2 is basically determined by fluctuation processes, i.e., for phase velocities less than or of the order of the thermal velocities, the oscillations have a purely fluctuation character. In that case $u_0^2 \ll 1$. Calculation shows that the frequencies of these oscillations are much less than the Langmuir frequency $\omega_0 = (4\pi n_0 e^2/m)^{1/2}$, their wavelength much less than the Debye screening radius λ_D and the energy density of the electrostatic field much less than the energy density of the thermal motion of the electrons. When $u_{ph} \gg 1$ we get from (13)

$$u_0^2 \approx u_{ph}^2(1 - A/u_{ph}). \quad (15)$$

The coefficient for the fluctuation correction is dropped since the magnitude of A is defined up to a factor of that order.

One can linearize Eq. (6) when $E_0 \ll 1$. It goes over into the equation for harmonic oscillations with a dispersion relation

$$\omega^2 = \omega_0^2(1 + 3T/mv_{ph}^2),$$

which is practically the same as the well known results of the linear theory. For the maximum field we get from (8) and (9)

$$E_{0\,max}^2/8\pi \approx n_0 m v_{ph}^2/2. \quad (16)$$

The ratio of the terms omitted in the last equation to those retained is of the order of magnitude $(T/mv_{ph}^2)^{1/4}$.

Electrostatic waves of large amplitude (the term "large amplitude" must here and in the following be taken in the sense of $E_0^2/n_0T \gg 1$) can thus be propagated in a plasma if the phase velocity of the wave is much larger than the thermal velocity of the electrons. One can neglect the influence of fluctuation processes on these waves. Physically this result is completely understandable. For processes which take place with velocities larger than the thermal velocity of the electrons, one can consider the plasma as a collective medium, the behavior of which is completely determined by the average value of the density n_0, the temperature T and the law of interaction between the particles.

It is not difficult to generalize this to the case of the propagation of waves in a plasma of moving beams. We shall again give our considerations in the system of coordinates in which the wave is at rest. We shall assume for the sake of simplicity that the temperatures of the beams are the same. We take as the equilibrium distribution functions of the beams Maxwell distributions. The statement of the problem and the solutions of the transport equations for the beams are exactly the same as before.

The Laplace equation in dimensionless quantities is of the form

$$d^2z/d\zeta^2 = \alpha_1\int_{u_0}^{\infty} F_1(t)(t^2+z)^{-1/2}t\,dt \Big/ \int_{u_0}^{\infty} F_1(t)\,dt$$
$$+ \alpha_2\int_{u_0}^{\infty} F_2(t)(t^2+z)^{-1/2}t\,dt \Big/ \int_{u_0}^{\infty} F_2(t)\,dt - 1,$$
$$F_{1,2}(t) = \exp[-(t-u_{1,2})^2] + \exp[-(t+u_{1,2})^2] \quad (17)$$

and together with the notation (5) we have introduced the new symbols

$$\alpha_1 = n_1/n_0,\ \alpha_2 = n_2/n_0,\ u_{1,2}^2 = m(v_{1,2} + v_{ph})^2/2T,$$

where n_1 and n_2 are the equilibrium densities of the first and the second beam; v_1 and v_2, there average velocities in the laboratory system of coordinates. Positive values of v_1 and v_2 correspond to a motion of the beams against the wave, i.e., u_1 and u_2 characterize the relative phase velocities for the first and the second beam.

The condition for the existence of stationary solutions of Eq. (7) is the condition for electrical neutrality of the plasma: $\alpha_1 + \alpha_2 = 1$. Integrating (17) twice leads to

$$\zeta = \int_0^z \frac{dz}{\sqrt{\widetilde{E}_0^2 - 2W(z)}}, \tag{18}$$

$$W(z) = z - 2\alpha_1 \int_{u_0}^{\infty} F_1(t)\left(\sqrt{t^2+z} - t\right) t\,dt \Big/ \int_{u_0}^{\infty} F_1(t)\,dt$$
$$- 2\alpha_2 \int_{u_0}^{\infty} F_2(t)\left(\sqrt{t^2+z} - t\right) t\,dt \Big/ \int_{u_0}^{\infty} F_2(t)\,dt. \tag{19}$$

We shall put, as before, the quantity u_0^2 equal to the sum of the average value $\overline{u^2}$ and a fluctuation correction for a particle described by the distribution function

$$f(u)\,du = [\alpha_1 \pi^{-1/2} \exp\{-(u-u_1)^2\}$$
$$+ \alpha_2 \pi^{-1/2} \exp\{-(u-u_2)^2\}]\,du, \tag{20}$$

i.e.,

$$u_0^2 = \overline{u^2}\left(1 - A\sqrt{\overline{(\Delta u^2)^2}/\overline{u^2}}\right).$$

We get from (20)

$$\overline{(\Delta u^2)^2} = \tfrac{1}{2} + 2\alpha_1 u_1^2 + 2\alpha_2 u_2^2 - \alpha_1^2 u_1^4 - \alpha_2^2 u_2^4 + \alpha_1 u_1^4$$
$$+ \alpha_2 u_2^4 - 2\alpha_1\alpha_2 u_1^2 u_2^2. \tag{21}$$

Solutions describing stationary processes of large amplitude with characteristics that depend weakly on the fluctuations are of interest. For them the conditions

$$\overline{u^2} \gg 1, \qquad \sqrt{\overline{(\Delta u^2)^2}}/\overline{u^2} \ll 1 \tag{22}$$

must be satisfied. We shall assume that $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $v_1 \neq v_2$. These cases go over into the one considered earlier. Let $\alpha_1 u_1^2$ be the larger of the quantities $\alpha_1 u_1^2$ and $\alpha_2 u_2^2$; we can then write the conditions (22), using (21), in the form

$$\alpha_1 u_1^2 \gg 1, \qquad \sqrt{\alpha_2/\alpha_1}\,|1 - u_2^2/u_1^2| \ll 1. \tag{23}$$

For beams with equal density we obtain the following conditions for the propagation of large amplitude waves:

$$m(v_1 - v_2)^2/2T \gg 1, \quad v_{ph} \approx -(v_1 + v_2)/2. \tag{24}$$

In particular, there can exist a standing wave with large amplitude in two beams going in opposite directions with the same velocity. If the densities in the beams are very different there is only one condition for propagation:

$$m(v_1 + v_{ph})^2/2T \gg 1,$$

where v_1 is the velocity of the denser beam.

The maximum field for which stationary processes are possible can be determined from (19). An estimate for $u_1 \gg 1$ and $u_2 \gg 1$ gives

$$E_{0\,max}^2/8\pi = \tfrac{1}{2} n_1 m (v_1 + v_{ph})^2 + \tfrac{1}{2} n_2 m (v_2 + v_{ph})^2. \tag{25}$$

The character of the stationary processes occurring in a plasma at rest or in motion is thus determined by the relative phase velocity. If the latter is equal to zero the field of a perturbation is screened by the electrons of low energy and is localized in a bounded region of space (in our considerations this corresponds to $u_0 = 0$ and $u_{ph} = 0$). When the phase velocity increases the number of electrons which can screen the field of the wave diminishes and for phase velocities much larger than the thermal velocities the screening action of the electrons can in general be neglected. The maximum fields which can then occur are determined by the relative phase velocity of the wave. The order of their magnitude is given by (16) and (25).

In a plasma of two beams of comparable density with different velocities there exist waves of two kinds. One has a phase velocity equal to the arithmetic mean of the values of the beam velocities. It is propagated along the faster beam. The condition for its propagation is a large relative velocity of the beams. The phase velocity of the second beam must be much larger than the velocity of either beam. With regard to this wave the beams may be considered to be a plasma at rest.

For small fields, waves with a large phase velocity have a harmonic character and their frequency is practically the same as the plasma frequency ω_0, while the dispersion law of these waves is determined by the well-known relation from the linear theory.

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