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INTEGRAL TRANSFORMATIONS OF THE I. S. SHAPIRO TYPE FOR PARTICLES OF ZERO MASS

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An expansion in terms of the irreducible representations of the proper Lorentz group is given for the representation which specifies the transformation of the wave function of a particle of zero mass and of arbitrary spin.

THE correspondence

$$\Psi(\mathbf{p},\sigma)\xrightarrow{S}\Psi'(\mathbf{p},\sigma)=\exp\{i\sigma\varphi(S,\mathbf{k})\}\Psi(S^{-1}\mathbf{p},\sigma),\qquad(1)$$

where S is a transformation belonging to the Lorentz group, $\mathbf{p}$ transforms like the momentum vector of a particle of mass 0, $\mathbf{k}=\mathbf{p}/p$, σ is an integer or a half-integer, and $\varphi(S,\mathbf{k})$ is the angle defined by formula (1.9) in reference 1 (hereafter referred to as I) has, as can be easily seen, the group property.

It defines the transformation law for the wave function of a particle of mass zero, and with spin component σ along the direction of the momentum $\mathbf{p}$, under transformations of the proper Lorentz group.[2]

The object of the present paper is to give an expansion of this representation in terms of the irreducible (ρ, m)-representations (cf. I) of the proper Lorentz group.

1. INTEGRAL TRANSFORMATIONS FOR PARTICLES OF MASS 0

In a manner similar to the way this was done in I we obtain the following system of mutually inverse integral transformations

$$\Psi(\mathbf{p},\sigma)=\int d\rho\int d\Omega(\mathbf{n})\,Y_{\rho m\mathbf{n}}(\mathbf{p},\sigma)\,f_{\rho m\mathbf{n}},\qquad(2)$$

$$f_{\rho m\mathbf{n}}=\sum_{\sigma}\int\frac{d^3\mathbf{p}}{|\mathbf{p}|}\,Y'_{\rho m\mathbf{n}}(\mathbf{p},\sigma)\,\Psi(\mathbf{p},\sigma),\qquad(3)$$

where $f_{\rho m\mathbf{n}}$ transforms according to the irreducible representation (ρ, m) of the proper Lorentz group, and we obtain the following conditions for determining the functions Y and Y':

$$Y_{\rho m S^{-1}\mathbf{n}}(S^{-1}\mathbf{p},\sigma)=\exp\{im\varphi(S,\mathbf{n})-i\sigma\varphi(S,\mathbf{k})\}\,[K(\mathbf{n})/K(S^{-1}\mathbf{n})]^{-1-i\rho/2}\,Y_{\rho m\mathbf{n}}(\mathbf{p},\sigma),\qquad(4)$$

where $K(\mathbf{n})$ is defined by formula (1.9), I. An analogous condition for $Y'_{\rho m\mathbf{n}}(\mathbf{p},\sigma)$ is satisfied if we take

$$Y'_{\rho m\mathbf{n}}(\mathbf{p},\sigma)=C_{m\rho}\overline{Y}_{\rho m\mathbf{n}}(\mathbf{p},\sigma).$$

Both here and later a bar above a letter denotes taking the complex conjugate.

We note that the function

$$Y_{\rho m\mathbf{n}}(\mathbf{p},\sigma)=(1/2\pi)\,\delta_{m\sigma}\,\delta(1-(\mathbf{nk}))\,p^{-1+i\rho/2}\qquad(5)$$

satisfies expression (4).

Thus, from (2) — (5) we obtain

$$\Psi(\mathbf{p}\ m)=\int d\rho\int d\Omega(\mathbf{n})\,\delta(\mathbf{n}-\mathbf{k})\,p^{-1+i\rho/2}f_{\rho m\mathbf{n}},\qquad(6)$$

$$f_{\rho m\mathbf{n}}=C_\rho\int\frac{d^3\mathbf{p}}{|\mathbf{p}|}\,\delta(\mathbf{n}-\mathbf{k})\,p^{-1-i\rho/2}\,\Psi(\mathbf{p},m).\qquad(7)$$

Here we have taken into account the fact that $\delta[1-(\mathbf{nk})]=2\pi\delta(\mathbf{n}-\mathbf{k})$.

On substituting (7) into (6) we find that $C_\rho = 1/4\pi$ while the integral over ρ in (6) should be taken from $-\infty$ to $+\infty$. Thus, the formulas

$$\Psi(\mathbf{p},m)=\int_{-\infty}^{+\infty}d\rho\int d\Omega(\mathbf{n})\,\delta(\mathbf{n}-\mathbf{k})\,p^{-1+i\rho/2}f_{\rho m\mathbf{n}},\qquad(8)$$

$$f_{\rho m\mathbf{n}}=\frac{1}{4\pi}\int\frac{d^3\mathbf{p}}{|\mathbf{p}|}\,\delta(\mathbf{n}-\mathbf{k})\,p^{-1-i\rho/2}\,\Psi(\mathbf{p},m)\qquad(9)$$

give a solution of the proposed problem; it turns out to be simpler than in the case of non-zero rest mass.*

2. COMPARISON OF THE RESULTS OBTAINED HERE WITH THOSE OF I

We compare the results obtained here with formulas (4.1), I and (4.2), I for M = 0 (M is the particle mass). A direct transition to the limit $M \to 0$ in the formulas indicated above is impossible. Instead of this we shall carry out the following formal manipulation of those formulas: we

*In the case m = 0 the same representation $(\rho, 0)$ is in fact contained twice in the result obtained, since the representations $(\rho, 0)$ and $(0, \rho)$ are equivalent. The transition between these two representations is given by formulas (14) and (15) with m = 0.

carry out the transition to the limit $M \to 0$ in the factor $R(L_p, n)$, we introduce a new variable of integration p/M (retaining for it the old notation p) and we replace the factor $\sqrt{1+p^2} - p \cdot n$ by $p - p \cdot n$. If we now introduce the components of the wavefunction having a definite component of the spin along the direction of the momentum we shall obtain

$$\tilde{\Psi}(p, m) = \int_{-\infty}^{0} d\rho \int d\Omega(n) \times [p - pn]^{-1+i\rho/2} e^{im\theta(p, n)} (-)^{s+m} \tilde{\varphi}_{-\rho-mn}, \quad (10)$$

$$\tilde{\varphi}_{-\rho-mn} = \frac{\rho^2 + (2m)^2}{(4\pi)^3} \int \frac{d^3p}{|p|} \times [p - pn]^{-1-i\rho/2} e^{-im\theta(p, n)} (-)^{s+m} \tilde{\Psi}(p, m). \quad (11)$$

Here

$$\tilde{\Psi}(p, m) = \sum_{\sigma} D^s_{\sigma m}(k) \tilde{\Psi}'(p, \sigma), \quad (12)$$

where $\tilde{\Psi}'(p, \sigma)$ is the wavefunction utilized in I; the notation $D^S_{\sigma m}(k)$ is also explained there. In the derivation of formulas (10) and (11) the following equation was used:

$$[D^s(k)^{-1} D^s(L_p n) D^s(n)]_{mn} \xrightarrow[M \to 0]{} \delta_{m, -n} (-1)^{s+m} e^{im\theta(p, n)}, \quad (13)$$

where $\theta(p, n)$ is defined in Appendix B.

We shall compare the expressions (8) and (9), obtained above with (10) and (11).* From the function $f_{\rho mn}$ which transforms according to the (ρ, m) representation we go over to the function $\varphi_{-\rho-mn}$, which transforms according to the $(-\rho, -m)$ representation. We make use of the fact that the representations (ρ, m) and $(-\rho, -m)$ are equivalent. Therefore the function $\varphi_{-\rho,-m,n}$ may be obtained from the function $f_{\rho mn}$ by the following unitary transformation:

$$\varphi_{-\rho, -m, n} = \int U_{\rho m}(n, k) f_{\rho m k} \, d\Omega(k), \quad (14)$$

$$f_{\rho m k} = \int \bar{U}_{\rho m}(n, k) \varphi_{-\rho, -m, n} \, d\Omega(n), \quad (15)$$

where

$$U_{\rho m}(n, k) = \sum_{l \geq |m|} \sum_{\alpha=-l}^{l} \frac{2l+1}{4\pi} \times \frac{\Gamma(l+1+i\rho/2)}{\Gamma(l+1-i\rho/2)} \bar{D}^l_{\alpha, -m}(n) D^l_{\alpha m}(k). \quad (16)$$

*It is proved later that (10) and (11) are not equivalent to (8) and (9) (and are therefore incorrect). We emphasize that this by no means indicates that the results of the present paper contradict those of I; it merely means that the formal manipulation which leads to (10) and (11) is not justified.

The function U is discussed in Appendix A and satisfies the following unitarity condition:*

$$\int \bar{U}_{\rho m}(l, n) U_{\rho m}(l, k) \, d\Omega(l) = \delta(n - k). \quad (17)$$

On substituting (14) and (15) into (8) and (9) we obtain

$$\varphi_{-\rho, -m, n} = \frac{1}{4\pi} \int \frac{d^3p}{|p|} p^{-1-i\rho/2} U_{\rho m}(n, k) \Psi(p, m), \quad (18)$$

$$\Psi(p, m) = \int_{-\infty}^{\infty} d\rho \int d\Omega(n) \bar{U}_{\rho m}(n, k) p^{-1+i\rho/2} \varphi_{-\rho, -m, n}. \quad (19)$$

We substitute into (18) and (19) the following formula derived in Appendix A:

$$U_{\rho m}(n, k) = A_{\rho m} [1 - (nk)]^{-1-i\rho/2} Q_m(n, k), \quad (20)$$

where

$$A_{\rho m} = 2^{1+i\rho/2} \Gamma(m + 1 + i\rho/2) / 4\pi \Gamma(m - i\rho/2), \quad (21)$$

$$Q_m(n, k) = m \sum_{l \geq |m|} \sum_{\alpha=-l}^{l} \frac{2l+1}{l(l+1)} \bar{D}^l_{\alpha, -m}(n) D^l_{\alpha m}(k). \quad (22)$$

We then obtain

$$\varphi_{-\rho, -m, n} = \frac{1}{4\pi} \int \frac{d^3p}{|p|} p^{-1-i\rho/2} A_{\rho m} \times [1 - (nk)]^{-1-i\rho/2} Q_m(n, k) \Psi(p, m), \quad (23)$$

$$\Psi(p, m) = \int_{-\infty}^{\infty} d\rho \int d\Omega(n) p^{-1+i\rho/2} \bar{A}_{\rho m} \times [1 - (nk)]^{-1+i\rho/2} \bar{Q}_m(n, k) \varphi_{-\rho, -m, n}. \quad (24)$$

Since

$$|A_{\rho m}|^2 = \frac{1}{(4\pi)^2} (\rho^2 + 4m^2)$$

and

$$Q_m(n, k) = e^{-im\theta(n, k)} (-1)^{s+m} \quad (25)$$

(cf. Appendix C), (23) and (24) differ from (10) and (11) only in that the integral over ρ in (23) and (24) is taken between the limits from $-\infty$ to ∞.

In particular, for $m = 0$ each irreducible representation $(\rho, 0)$ occurs twice in the expansion under consideration in contrast to the case $M \neq 0$.

APPENDIX A

DEFINITION OF THE FUNCTION $U_{\rho m}(n, k)$

It may be easily shown that from the fundamental relations (14), (15) and the transformation law for $f_{\rho m k}$ and $\varphi_{-\rho,-m,n}$ the following functional equation for $U_{\rho m}(n, k)$ may be obtained:

*In particular, for $m = 0$ we obtain the following simple integral representation for the δ-function:

$$\frac{\rho^2}{(4\pi)^2} \int [1 - (ln)]^{-1+i\rho/2} [1 - (lk)]^{-1-i\rho/2} d\Omega(l) = \delta(n - k)$$

[cf. also formulas (A.4) and (A.6)].

$$U_{\rho m}(S^{-1}\mathbf{n}, S^{-1}\mathbf{k}) = U_{\rho m}(\mathbf{n},\mathbf{k})[K(\mathbf{n})K(\mathbf{k})/K(S^{-1}\mathbf{n})K(S^{-1}\mathbf{k})]^{-1-i\rho/2} \times \exp\{im[\varphi(S,\mathbf{n})+\varphi(S,\mathbf{k})]\} \quad (A.1)$$

(the notations $K(\mathbf{n})/K(S^{-1}\mathbf{n})$ and $\varphi(S,\mathbf{n})$ are defined in I). Since the functions $D^l_{\alpha m}(\mathbf{k})$ for $l = |m|, |m|+1, \ldots$ and for fixed m form a complete system, U may be represented in the following form

$$U_{\rho m}(\mathbf{n}, \mathbf{k}) = \sum_{l\alpha}\sum_{l'\beta} X_{l\alpha l'\beta}\overline{D}^l_{\alpha,-m}(\mathbf{n})\, D^{l'}_{\beta m}(\mathbf{k}). \quad (A.2)$$

On taking in formula (A.1) for S the pure rotation $S = R$ we obtain on taking into account formula (1.9b), I,

$$X_{l\alpha l'\beta} = X_l\delta_{ll'}\delta_{\alpha\beta}.$$

Further, we take in (A.1) for S the infinitesimal pure Lorentz transformation L:

$$L^{-1}\mathbf{n} = (\mathbf{n}+\boldsymbol{\varphi})/[1+(\mathbf{n}\boldsymbol{\varphi})].$$

It can then be easily seen that

$$K(L^{-1}\mathbf{n})/K(\mathbf{n}) = 1 + (\boldsymbol{\varphi}\mathbf{n}).$$

Since according to (1.9d), I

$$e^{im\varphi(L,\mathbf{k})}D^l_{\beta m}(L^{-1}\mathbf{k}) = \sum_\gamma D^l_{\beta\gamma}(R(L,\mathbf{k}))\, D^l_{\gamma m}(\mathbf{k})$$

we obtain from (A.1) and (A.2)

$$\sum X_l\overline{D}^l_{\alpha-m}(\mathbf{n})\, D^l_{\alpha m}(\mathbf{k}) = [K(\mathbf{n})K(\mathbf{k})/K(L^{-1}\mathbf{n})K(L^{-1}\mathbf{k})]^{1+i\rho/2} \times \sum X_l\overline{D}^l_{\alpha\gamma}(R(L,\mathbf{n}))\,\overline{D}^l_{\gamma-m}(\mathbf{n})\, D^l_{\alpha\delta}(R(L,\mathbf{k}))\, D^l_{\delta m}(\mathbf{k}). \quad (A.3)$$

The parameter of the rotation R occurring in the above is defined by formula (A.4), I:

$$D^l(R(L,\mathbf{k})) = e^{-i(\mathbf{H}\boldsymbol{\alpha})} \approx 1 - i(\mathbf{H}\boldsymbol{\alpha}), \qquad \boldsymbol{\alpha} = [\mathbf{k}\times\boldsymbol{\varphi}].$$

We then obtain from formula (A.3)

$$\sum X_l\overline{D}^l_{\alpha,-m}(\mathbf{n})\, D^l_{\alpha m}(\mathbf{k}) = [1-(1+i\rho/2)\boldsymbol{\varphi}(\mathbf{n}+\mathbf{k})] \times \sum X_l\{\overline{[1-i(\mathbf{H}^l[\mathbf{n}\times\boldsymbol{\varphi}])]\, D^l(\mathbf{n})}\}_{\alpha,-m} \times \{[1-i(\mathbf{H}^l[\mathbf{k}\times\boldsymbol{\varphi}])]\, D^l(\mathbf{k})\}_{\alpha,m}.$$

Here we must express the cyclic components of the vectors $\mathbf{n}$ and $\mathbf{k}$ in terms of the generalized spherical harmonics $D^1_{\alpha 0}(\mathbf{n})$, $D^1_{\alpha 0}(\mathbf{k})$, and we must then eliminate products of the D-functions in accordance with the following rule

$$D^1_{ab}(\mathbf{n})\, D^l_{cd}(\mathbf{n}) = \langle 1lac\,|\,LM\rangle\,\langle 1l\,bd\,|\,LN\rangle\, D^L_{MN}(\mathbf{n}).$$

By equating to zero the coefficient of φ we obtain (in the intermediate steps of the calculation we make use of Racah's rule for combining three Clebsch-Gordan coefficients into one):

$$X_l[l(l+1) - l'(l'+1) - i\rho] + (-)^{l-l'+1}X_{l'}[l'(l'+1) - l(l+1) - i\rho](2l+1)/(2l'+1) = 0.$$

From this it follows that

$$X_l = C(2l+1)\,\Gamma(l+1+i\rho/2)/\Gamma(l+1-i\rho/2).$$

On utilizing the unitarity condition (17) already mentioned in the main text we obtain, finally, formula (16).

In order to obtain formula (20) we note that the function

$$Q_{\rho m}(\mathbf{n}, \mathbf{k}) \equiv [1-(\mathbf{nk})]^{1+i\rho/2}\, U_{\rho m}(\mathbf{n}, \mathbf{k}) \quad (A.4)$$

satisfies the same functional equation (A.1) which is satisfied also by $U_{\rho m}(\mathbf{n}, \mathbf{k})$, only we must set in it $1 + i\rho/2 = 0$. From this it follows that

$$Q_{\rho m}(\mathbf{n}, \mathbf{k}) = A_{\rho m}\sum_{l\geqslant|m|}\frac{2l+1}{l(l+1)}\overline{D}^l_{\alpha-m}(\mathbf{n})\, D^l_{\alpha m}(\mathbf{k}).$$

In order to find A_ρ, we set $\mathbf{k} = -\mathbf{n}$ in (A.4).

Since

$$D^l(\mathbf{n}) = R_3(\varphi+\pi/2)\, R_1(\theta),$$
$$D^l(-\mathbf{n}) = R_3(\varphi+3\pi/2)\, R_1(\pi-\theta),$$

then

$$[D^l(\mathbf{n})^{-1}D^l(-\mathbf{n})]_{-m,m} = [R_1(-\theta)\, R_3(\pi)\, R_1(\pi-\theta)]^l_{-m,m} = [R_1(-\pi)\, R_3(\pi)]^l_{-m,m} = e^{-im\pi}(-1)^{2l}(-1)^{l+m}e^{-im\pi}.$$

Since

$$\sum_{l\geqslant|m|}\frac{2l+1}{l(l+1)}(-1)^{l+m} = \sum_{l\geqslant|m|}\left(\frac{1}{l+1}(-1)^{l+m} + \frac{1}{l}(-1)^{l+m}\right) = (-1)^{2m}\frac{1}{m}$$

and, moreover,

$$\sum_{l\geqslant|m|}(-1)^{l+m}(2l+1)\,\Gamma(l+1+i\rho/2)/\Gamma(l+1-i\rho/2) = \sum_{l\geqslant m}(-1)^{l+m}\left[\frac{\Gamma(l+2+i\rho/2)}{\Gamma(l+1-i\rho/2)} + \frac{\Gamma(l+1+i\rho/2)}{\Gamma(l-i\rho/2)}\right] = (-1)^{2m}\,\Gamma(m+1+i\rho/2)/\Gamma(m-i\rho/2),$$

the proof of formula (20) is complete.

APPENDIX B

PROOF OF FORMULA (13)

According to formula (1.9b), I,

$$D^S(L_\mathbf{p}\mathbf{n})\, D^S(\mathbf{n}) = D^S(L_\mathbf{p}^{-1}\mathbf{n})\, R_3(\varphi(L_\mathbf{p}, \mathbf{n})).$$

Further, from formula (A.2) of Appendix A in I it may be easily seen that

$$L_\mathbf{p}^{-1}\mathbf{n} \xrightarrow[M\to 0]{} -\mathbf{k}.$$

Finally, we have,

$$[R(\mathbf{k})]^{-1} R(-\mathbf{k})$$

$$= R_1(-\theta) R_3(-\varphi - \pi/2) R_3(\varphi - \pi + \pi/2) R_1(\pi - \theta)$$

$$= R_3(-\pi) R_1(\pi),$$

$$[D^S(\mathbf{k})^{-1} D^S(-\mathbf{k})]_{mn} = [R_3(-\pi) R_1(\pi)]^S_{mn} = \delta_{n,-m}(-1)^{S+m}.$$

Therefore,

$$\theta(\mathbf{p}, \mathbf{n}) = \lim_{M \to 0} \varphi(L_{\mathbf{p}}, \mathbf{n}). \qquad \text{(B.1)}$$

APPENDIX C

PROOF OF FORMULA (25)

In formula (A.1) we set $1 + i\rho/2 = 0$, $S = L_{\mathbf{p}}$, with $\varphi(S, \mathbf{k}) = 0$. Further, we let M tend to zero; we then have

$$S^{-1}\mathbf{k} \to \mathbf{k}, \quad S^{-1}\mathbf{n} \to -\mathbf{k},$$

$$Q_m(S^{-1}\mathbf{n}, S^{-1}\mathbf{k}) \to Q_m(-\mathbf{k}, \mathbf{k}) = (-1)^{S+M}.$$

Thus from (A.5) and (B.1) we obtain formula (25).

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[1] Chou Kuang-Chao and L. G. Zastavenko, JETP **35**, 1417 (1958), Soviet Phys. JETP **8**, 990 (1959).

[2] Chou Kuang-Chao, JETP **36**, 909 (1959), Soviet Phys. JETP **9**, 642 (1960).

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