A METHOD FOR THE CALCULATION OF PHASE VOLUMES

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A method for calculating phase volumes is developed for 2, 3, 4, and 5 particles. The error of the method increases with the number n of particles, but evidently does not exceed 5 percent for n = 5.

1. INTRODUCTION

PHASE volume is the name given to the quantity

$$\rho(E, M_1, M_2, \ldots, M_n) = \int d^3p_1 \int d^3p_2 \ldots \int d^3p_n \times \delta(\mathbf{p}_1 + \mathbf{p}_2 + \ldots \mathbf{p}_n)\, \delta\left(\sum_{i=1}^{n} \sqrt{M_i^2 + p_i^2} - E\right), \qquad (1)$$

which is a function of the total energy E of the particles and of their masses M_i.

A knowledge of the phase volumes is needed in the Fermi statistical theory, so that methods for their calculation are dealt with in a number of papers. Fermi[1] calculated phase volumes for the case in which some of the particles are nonrelativistic and some are ultrarelativistic, with conservation of momentum not taken into account for the latter particles. In a paper by Maksimenko and Rozental'[2] these same phase volumes have been calculated with conservation of the total momentum taken into account.

Papers by Lepore and Stuart[3] and by Rozental'[4] give the expression for the phase volume when all the particles are ultrarelativistic (with total momentum conserved). Lepore and Stuart[3] have shown that the phase volume (1) can be represented in the form

$$\rho = \frac{4\pi (2\pi^2)^n \prod M_\alpha^2}{(2\pi)^4} \int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} d\lambda e^{i\lambda E} \lambda^n \times \int_{-\infty}^{+\infty} \sigma^2 d\sigma \frac{\prod_{\beta=1}^{n} H_2^{(1)} \left[-M_\beta \sqrt{\lambda^2 - \sigma^2}\right]}{[\sigma^2 - \lambda^2]^n}, \qquad (2)$$

from which an expansion of ρ in powers of $1/E$ has been obtained,[2] useful in principle for the calculation of any phase volume, but which is inconvenient on account of its complication.

A paper by Fialho[5] proposes a simple method for calculating phase volumes by starting with Eq. (2) and using the method of steepest descents; this approach is useful for all energies E. We shall give a different method for calculating phase volumes, which is more accurate and convenient when the number n of particles involved is small.

2. THE PROPOSED METHOD

Our method is to calculate the function of the masses, $\rho(E, M_1, \ldots, M_n)$, for certain values of the masses M_i and then obtain the whole function of the variables $M_1, \ldots, M_n$ by interpolation.

Let us define a function $f(E, M_1, \ldots, M_n)$ by the formula

$$\rho(E, M_1, \ldots, M_n) = A_n E^{3n-4} f(E, M_1, \ldots, M_n), \quad A_n = \frac{(\pi/2)^{n-1} (4n-4)!}{(3n-4)!\,(2n-1)!\,(2n-2)!}. \qquad (3)$$

It follows from Eq. (1) that f is a homogeneous function of degree zero in its arguments. Thus if

$$x = E / \left(\sum_{\alpha=1}^{n} M_\alpha\right), \quad z_i = M_i/(\Sigma M_\alpha); \quad \sum_{i=1}^{n} z_i = 1,$$

then

$$f(E, M_1, \ldots, M_n) = f(x, z_1, \ldots, z_n). \qquad (4)$$

By the formulas of Maksimenko and Rozental',[2] and also by formulas obtained in the Appendix, we have calculated tables of the functions ${}_nf_k(x)$:

$${}_nf_1(x) = f(x, 1, 0, \ldots, 0),$$

$${}_nf_2(x) = f(x, \tfrac{1}{2}, \tfrac{1}{2}, 0, \ldots, 0), \ldots,$$

$${}_nf_n(x) = f(x, 1/n, 1/n, \ldots, 1/n).$$

In order to get an approximate construction of the whole function $f(x, z_1, \ldots, z_n)$ from the ${}_nf_k(x)$,

$k = 1, 2, \ldots, n$, we have used the coefficients*

$$_nC_k(z_1, z_2, \ldots, z_n) = \sum_{s=k}^{n} {}_sA_k \sum_{i_1>i_2\ldots>i_s} z_{i_1}z_{i_2}\ldots z_{i_s},$$

$$_sA_k = (-)^{s-k} s k^{s-1} (s!) [(k!)(s-k)!]^{-1}, \qquad (5)$$

which have the following properties which are basic for our interpolation:

$$_nC_k(1, 0, 0, \ldots, 0) = 0,$$

$$_nC_k\left(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\right) = 0, \ldots$$

$$\ldots \text{and only } {}_nC_k(\overbrace{1/k, 1/k, \ldots, 1/k}^{\text{k times}}, 0, \ldots, 0) = 1 \qquad (6)$$

and which also satisfy the relation:

$$\text{if } \Sigma z_i = 1,$$

then

$$\sum_{k=n}^{n} {}_nC_k(z_1, z_2, \ldots, z_n) = 1. \qquad (7)$$

As an approximation to the phase volume we take the function

$$\tilde{\rho}(E, M_1, \ldots, M_n) = A_n E^{3n-4}$$

$$\times \left\{ \sum_{k=1}^{n} {}_nC_k(z_1, \ldots, z_n) [{}_nf_k(x)]^{2/3} \right\}^{3/2}. \qquad (8)$$

Here z_i and x are expressed in terms of M_i and E by (4), the functions $[{}_nf_k(x)]^{2/3}$ are found from Table I in reference 8, the coefficients $_nC_k(z_1, \ldots, z_n)$ are defined by (5), and the coefficient A_n is defined by (3).

In virtue of (7) and the fact that for $x \to \infty$ all the functions $_nf_k(x)$ become the same, in the ultrarelativistic case $\tilde{\rho}$ no longer depends on the masses, as should be true of the phase volume. Since for $x \to 1$ the functions $_nf_n(x)$ are much larger than all the others, by Eq. (5) the function $\tilde{\rho}$ has the right dependence on the masses of the particles in the nonrelativistic case:

$$\tilde{\rho} \sim \left(\prod_{\alpha=1}^{n} M_\alpha \right)^{3/2}$$

Thus our interpretation is confirmed by the agreement of $\tilde{\rho}$ with the known nonrelativistic and ultrarelativistic limits.

Equation (8) is indeed our final formula. We emphasize that the table of $[{}_nf_k(x)]^{2/3}$ in reference 8 has been constructed only for $n = 2, 3, 4, 5$.

*The writer is grateful to V. B. Magalinskiĭ for pointing out this unified formula for the $_nC_k$

To estimate the errors of the method we have made a comparison of the values of $\tilde{\rho}(E, M_1, M_2, \ldots)$ calculated from Eq. (6) with the exact values of the function $\rho(E, M_1, M_2, \ldots)$ found from the formulas of Maksimenko and Rozental' and of the Appendix, for certain values of E and M for 2, 3, and 5 particles (cf. Table II in reference 8).

The comparison gives a basis for the supposition that if the masses of any two particles do not differ by more than a factor 10, for numbers of particles up to and including 5 our method gives an error of not more than 5 percent (for $n = 2$ the largest error we have found in the method is 2.2 percent; it occurs when the mass m_1 of one of the particles goes to zero, and the total energy E goes to the rest mass m_2 of the second particle in such a way that $E - m_2 \approx 2m_1$).

In conclusion I express my deep gratitude to Professor M. A. Markov and to L. A. Chudov, V. I. Ogievetskiĭ, and G. I. Kopylov for the support they have given me and for a valuable consultation. The writer is also grateful to the staff of the computation laboratory directed by G. I. Kopylov for making the numerical calculations and to V. M. Maksimenko for making it possible for me to familiarize myself with his paper[2] before its publication.

APPENDIX

The expansions obtained by Maksimenko and Rozental'[2] were used for the calculation of the functions $_nf_k(x)$ for $x - 1 \gtrsim 1$. For $x - 1 \lesssim 1$ these expansions become inconvenient, and we found it necessary to obtain an expansion of the function ρ in powers of $x - 1$. For this purpose, in turn, we had to obtain a representation of the phase volume in the form of a single contour integral.

A. Transformation of the Integral (2)

In reference 2 the expression (2) is given in the form

$$\rho = \frac{\pi^{n-3}}{2^{n+2}} \left(\prod_{\alpha=1}^{n} M_\alpha^2 \right) \iint_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} dx dy e^{i(x+y)E}$$

$$\times \frac{(x-y)^2}{(xy)^n} i^n (x+y)^n \times \prod_{\alpha=1}^{n} \left\{ \frac{\pi}{i} H_2^{(2)}(2M_\alpha \sqrt{xy}) \right\}. \quad (A.1)$$

This expression can be written in the form (we write $\nu_\alpha = M_\alpha / E$)

$$\rho = \frac{\pi^{n-3}}{2^{n+2}} \left(\frac{d}{dE} \right)^n E^{4n-4} \left(\prod_{\alpha=1}^{n} \nu_\alpha^2 \right) \iint dx dy e^{i(x+y)} \frac{(x-y)^2}{(xy)^n}$$

$$\times \prod_{\alpha=1}^{n} \left\{ \frac{\pi}{i} H_2^2 (2\nu_\alpha \sqrt{xy}) \right\}. \qquad (A.2)$$

Substituting $x = t\tau$, $y = t/\tau$ in this formula, we get

$$U \equiv \iint dxdy \frac{e^{i(x+y)}}{(xy)^n} (x-y)^2 \prod_{\alpha=1}^{n} \left\{ \frac{\pi\nu_\alpha^2}{i} H_2^{(2)}(2\nu_\alpha \sqrt{xy}) \right\}$$

$$= \frac{8\pi}{i} 2^{2n-3} \int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} dt \frac{J_1(t)}{t^{2n-2}} \prod_{\alpha=1}^{n} \left\{ \frac{\pi}{i} \nu_\alpha^2 H_2^{(2)}(\nu_\alpha t) \right\}.$$

Thus we have

$$\rho(E, M_1, M_2, \ldots) = 2^{-n-2}\pi^{n-3}(d/dE)^n E^{3n-4} U, \quad (A.3)$$

and U contains only a single integral.

B. Expansion of the Function $\rho(E, M_1, M_2, \ldots)$ in Powers of $E-1$

We substitute in Eq. (A.3) the formula

$$-i\pi H_2^{(2)}(\nu t) = (2\pi/\nu t)^{1/2} \exp\{i(3\pi/4 - \nu t)\}_2 F_0 \times \left(\frac{5}{2}, -\frac{3}{2}; -1/2i\nu t\right),$$

where (cf. reference 6)

$$_2F_0\left(\frac{5}{2}, -\frac{3}{2}; x\right) = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{5}{2}+k\right)\Gamma\left(-\frac{3}{2}+k\right)}{\Gamma\left(\frac{5}{2}\right)\Gamma\left(-\frac{3}{2}\right)k!} x^k. \quad (B.1)$$

For the values of α for which $\nu_\alpha = 0$, we must take $\nu^2 H_2^{(2)}(\nu t) = 4t^{-2}$. We get

$$U = \frac{8\pi}{i} 2^{(5n+3l-6)/2} \pi^{(n-l)/2} e^{3\pi i m/4} \prod_{\alpha=1}^{m} \nu_\alpha^{3/2} \sum_{k_1, k_2, \ldots = 0}^{\infty} 2^{-k} e^{i\pi k/2}$$

$$\times \frac{a_{k_1} a_{k_2} \ldots a_{k_m}}{\nu_1^{k_1} \nu_2^{k_2} \ldots \nu_m^{k_m}} \int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} dt \cdot t^{-5n/2-k-3l/2} e^{-i\nu t} J_1(t),$$

$$\nu = \sum_{\alpha=1}^{m} \nu_\alpha, \quad k = \sum_{\alpha=1}^{m} k_\alpha. \quad (B.2)$$

Here l is the number of particles with mass zero, $m = n - l$, and a_k is the coefficient of x^k in Eq. (B.1). The integral in Eq. (B.2) is calculated by a procedure like that of reference 6, page 421; by use of quadratic identities for the hypergeometric functions it can be brought to the form

$$\int dt J_1(t) e^{-i\nu t} t^{2-k-5n/2-3l/2} = 2\pi i 2^{2-k-5n/2-3l/2}$$

$$\times \exp\left\{-i\frac{\pi}{2}\left(4-k-\frac{5n}{2}-\frac{3l}{2}\right)\right\} [2(1-\nu)]^{5n/2-5/2+k+3l/2}$$

$$\times \frac{F\left(\frac{3}{2}, -\frac{1}{2}; \gamma_k; (1-\nu)/2\right)}{\sqrt{\pi}\,\Gamma(\gamma_k)}. \quad (B.3)$$

Substituting the expression for U so obtained in Eq. (A.3), we find

$$\rho(E, M_1, M_2, \ldots, M_m, 0, 0, \ldots, 0) = 2^{2l}(2\pi)^{(3n-l-3)/2}$$

$$\times \left(\frac{d}{dE}\right)^n \left(\prod_{\alpha=1}^{m} M_\alpha^{3/2}\right) \sum \frac{a_{k_1} \ldots a_{k_m} (-)^k}{\nu_1^{k_1} \ldots \nu_m^{k_m} (2E)^k \Gamma(\gamma_k)}$$

$$\times \frac{(E-M)^{\gamma_k - 1}}{E^{3/2}} {}_2F_1\left(\frac{3}{2}, -\frac{1}{2}; \gamma_k; \frac{E-M}{2E}\right). \quad (B.4)$$

Here and in the preceding formula $\gamma_k = (5n + 3l - 3)/2 + k$.

To obtain formulas for the calculation of the functions we want we still have to perform the differentiation in Eq. (B.4). Let us define the numbers $\varphi_p^{(n)}$ by the relation

$$\left(\frac{d}{dE}\right)^n \frac{(E-M)^{\gamma-1}}{E^{3/2}} {}_2F_1\left(\frac{3}{2}, -\frac{1}{2}; \gamma; \frac{E-M}{2E}\right)$$

$$= \frac{(\gamma-1)(\gamma-2)\ldots(\gamma-n)}{\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\ldots\left(\frac{1}{2}+n\right)}$$

$$\times \sum_{p=0}^{\infty} \frac{\left(\frac{3}{2}\right)_p \left(-\frac{1}{2}-n\right)_p (E-M)^{\gamma+p-n}}{(\gamma-n)_p p!\, 2^p E^{p+3/2}} \varphi_p^{(n)},$$

$$(\alpha)_p \equiv \alpha(\alpha+1)\ldots(\alpha+p-1). \quad (B.5)$$

The $\varphi_p^{(n)}$ satisfy the obvious relation

$$2p\varphi_{p-1}^{(n)} - \left(p - n - \frac{3}{2}\right)\varphi_p^{(n)} = \varphi_p^{(n+1)}$$

and the relation

$$\varphi_{p+1}^{(n+1)} - \varphi_p^{(n+1)} = (n+1)\varphi_p^{(n)},$$

which is extremely convenient for their calculation.

Substituting Eq. (B.5) in Eq. (B.4), we get the final expression

$$\rho(E, M_1, M_2, \ldots M_m, 0, \ldots) = 2^{2l}(2\pi)^{[3(n-1)-l]/2}$$

$$\times \frac{(\Pi M_\alpha^{3/2})}{E^{3/2}} \frac{(E-M)^{[3(n+l)-5]/2}}{\Gamma([3(n+l)-3]/2)}$$

$$\times \sum \frac{(-)^k a_{k_1} \ldots a_{k_m} (E-M)^k \Gamma([3(n+l)-3]/2)}{2^k M_1^{k_1} \ldots M_m^{k_m} \Gamma([3(n+l)-3]/2+k)}$$

$$\times F_{nlk}\left(\frac{E-M}{2E}\right), \quad (B.6)$$

where

$$F_{nlk}(x) = \sum_{p=0}^{\infty} \frac{\left(\frac{3}{2}\right)_p \left(-\frac{1}{2}-n\right)_p \varphi_p^{(n)} x^p}{([3(n+l)-3]/2+k)_p\, p!\left(\frac{3}{2}\right)_p}$$

The basis functions ${}_n\rho_m(E)$ are obtained if in Eq. (B.6) we take $M_1 = M_2 = \ldots M_m = 1/m$.

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