EQUILIBRIUM OF A PLASMA TOROID IN A MAGNETIC FIELD

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The equilibrium conditions for a plasma toroid with a distributed current in a magnetic field are obtained.

1. INTRODUCTION

In the present work we derive exact solutions for the equilibrium of a toroidal pinch with axial symmetry. As has been shown earlier,[1] a bound, axially symmetric, plasma configuration can be in equilibrium in a magnetic field only if there is a nonvanishing azimuthal current component j_φ. It has also been shown that for equilibrium to obtain the current density j_φ must be of the form

$$j_\varphi(r, z) = Ar + B/r, \tag{1}$$

where r is the distance from the axis of symmetry and A and B are arbitrary functions which are constant over surfaces of equal plasma pressure. The plasma pressure p and the functions

$$\psi = \int^r H_z 2\pi r dr, \qquad I = \int_0^r j_z 2\pi r dr = crH_\varphi/2, \tag{2}$$

through which the r and z components of the magnetic field and current are expressed, are interdependent. The functions A and B are expressed in terms of ψ and I in the following way:

$$A(\psi) = 2\pi c \frac{dp(\psi)}{d\psi}, \qquad B(\psi) = \frac{1}{c}\frac{dI^2(\psi)}{d\psi}. \tag{3}$$

The function ψ, which is related to the azimuthal component of the vector potential of the magnetic field A_φ, is given by the following equation (in cylindrical coordinates):

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = -\frac{8\pi^2}{c} r j_\varphi = -\frac{8\pi^2}{c}[A(\psi) r^2 + B(\psi)]. \tag{4}$$

Given the actual form of the functions $A(\psi)$ and $B(\psi)$, it is possible in principle to solve this equation and then, knowing $\psi(r, z)$, to find the distribution of all quantities which characterize equilibrium. In this case, if the geometric configuration is given the conditions at infinity (or at external conductors) cannot be chosen arbitrarily since they are determined by the solutions of the problem. On the other hand, if the conditions at infinity or at the external conductors are given the geometric configuration is no longer arbitrary. In practice, it is obviously more convenient to assign the cross section of the pinch in the r-z plane and to use the coordinate system most convenient for solution of the problem. Below we consider a pinch of circular cross section with fixed values of A and B.

2. TOROIDAL COORDINATES

The most convenient coordinate system is the usual toroidal coordinate system, defined by ϑ, ω, and φ:

$$r = R_0 \sinh\vartheta / (\cosh\vartheta - \cos\omega),$$
$$z = R_0 \sin\omega / (\cosh\vartheta - \cos\omega). \tag{5}$$

Here $\vartheta = \vartheta_0$ defines the cross section of the toroidal pinch. The small and large radii of the toroid are then

$$a = R_0/\sinh\vartheta_0, \qquad R = R_0 \coth\vartheta_0. \tag{6}$$

Values for which $\vartheta > \vartheta_0$ correspond to the inner part of the toroid while values for which $\vartheta < \vartheta_0$ correspond to the outer part. In toroidal coordinates the components of the magnetic field are expressed in terms of ψ as follows:

$$H_\vartheta = \frac{(\cosh\vartheta - \cos\omega)^2}{2\pi R_0^2 \sinh\vartheta}\frac{\partial\psi}{\partial\omega}, \quad H_\omega = -\frac{(\cosh\vartheta - \cos\omega)^2}{2\pi R_0^2 \sinh\vartheta}\frac{\partial\psi}{\partial\vartheta}. \tag{7}$$

The current density components j_ϑ, j_ω are expressed in terms of $I(\vartheta, \omega)$ in similar fashion.

As Fock has shown,[2] the variables can be separated in the homogeneous equation for the vector potential. Thus Eq. (4) can be solved easily if the right-hand side is independent of ψ, i.e., if

$$A = \text{const}, \qquad B = \text{const}. \tag{8}$$

This simple case is the one which is considered in the present paper.

If A and B are constant the particular solution of the inhomogeneous equation (4) is

$$\psi_1 = -(4\pi^2/c)(Ar^2 + B) z^2. \tag{9}$$

In order to find the general solution of the corresponding homogeneous equation, following Fock we introduce the auxiliary function $F(\vartheta, \omega)$ which is related to $\psi(\vartheta, \omega)$ by the expression

$$\psi(\vartheta, \omega) = F(\vartheta, \omega)/\sqrt{2(\coth\vartheta - \cos\omega)}. \quad (10)$$

In toroidal coordinates, Eq. (4) for the function F becomes

$$\frac{\partial^2 F}{\partial\vartheta^2} + \frac{\partial^2 F}{\partial\omega^2} - \coth\vartheta\frac{\partial F}{\partial\vartheta} + \frac{1}{4}F = -\frac{32\pi^2 R_0^2}{c\,[2(\cosh\vartheta - \cos\omega)]^{5/2}} \times\left[\frac{4AR_0^2\sinh^2\vartheta}{[2(\cosh\vartheta - \cos\omega)]^2} + B\right]. \quad (11)$$

The solution can be written as a Fourier series in ω:

$$F = F_0(\vartheta) + 2\sum_{n=1}^{\infty} F_n(\vartheta)\cos n\omega. \quad (12)$$

Following Fock we take the linearly independent solutions of the homogeneous equation for F to be*

$$g_n(\vartheta)\cos n\omega, \quad f_n(\vartheta)\cos n\omega, \quad (13)$$

$$g_n(\vartheta) = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{2(\cosh\vartheta - \cos\omega)}\cos n\omega\, d\omega.$$

$$f_n(\vartheta) = \frac{1}{2\pi}\int_{-\vartheta}^{\vartheta}\sqrt{2(\cosh\vartheta - \cosh t)}\cosh nt\, dt. \quad (15)$$

The Wronskian is

$$f_n\frac{dg_n}{d\vartheta} - g_n\frac{df_n}{d\vartheta} = \frac{\sinh\vartheta}{\pi(n^2 - 1/4)}. \quad (16)$$

The equation by which the functions f_n and g_n are obtained is

$$\frac{d^2 g_n}{d\vartheta^2} - \coth\vartheta\frac{dg_n}{d\vartheta} - \left(n^2 - \frac{1}{4}\right)g_n = 0. \quad (17)$$

The function $f_n(\vartheta)$ becomes infinite at the center of the toroid $\vartheta = \infty$, but is finite (as are its derivatives) in the outer region. Hence, we cannot use the function $g_n(\vartheta)$ in the inner region. Since it is impossible to satisfy the condition that the field vanish at infinity for the outer problem, we must use both linearly independent solutions.

We note that the particular solution for ψ in (9) corresponds to the following particular solution for F:

$$F_1 = -\frac{16\pi^2 R_0^2}{c}\frac{\sinh^2\omega}{[2(\cosh\vartheta - \cos\omega)]^{3/2}} \times\left[4AR_0^2\frac{\sinh^2\vartheta}{[2(\cosh\vartheta - \cos\omega)]^{3/2}} + B\right]. \quad (18)$$

*These functions are related to the Legendre functions $P_{n-1/2}$, $Q_{n-1/2}$ by the following expressions (cf. reference 2):

$$(n^2 - 1/4)f_n(\vartheta) = \sinh\vartheta\, dP_{n-1/2}(\cosh\vartheta)/d\vartheta,$$

$$(n^2 - 1/4)g_n(\vartheta) = \sinh\vartheta\, dQ_{n-1/2}(\cosh\vartheta)/d\vartheta.$$

We expand this solution in a Fourier series

$$-F_1 = \gamma_0(\vartheta) + 2\sum_{n=1}^{\infty}\gamma_n(\vartheta)\cos n\omega. \quad (19)$$

In place of the constants A and B, which determine the current density j_φ, we introduce the total volume current J_φ through the cross section $\vartheta > \vartheta_0$, and the partial current J_p:

$$J_\varphi = \int_{\vartheta>\vartheta_0} j_\varphi dr dz = \int_{\vartheta>\vartheta_0}(Ar + B/r)\,dr dz \quad (20)$$

$$= J_p + 2\pi B R_0/e^{\vartheta_0}\sinh\vartheta_0,$$

$$J_p = \int_{\vartheta>\vartheta_0} Ar dr dz = \pi A R_0^3\cosh\vartheta_0/\sinh^3\vartheta_0. \quad (21)$$

The Fourier coefficients $\gamma_n(\vartheta)$ in (19) are written in the form

$$\gamma_n(\vartheta) = \frac{8\pi}{c}R_0 J_p\frac{\sinh^3\vartheta_0}{\cosh\vartheta_0}\left[8\alpha_n(\vartheta) - \frac{\cosh\vartheta_0 e^{\vartheta_0}}{\sinh\vartheta_0}\beta_n(\vartheta)\right] + \frac{8\pi}{c}R_0 J_\varphi\sinh\vartheta_0 e^{\vartheta_0}\beta_n(\vartheta), \quad (22)$$

where

$$\beta_n(\vartheta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin^2\omega\cos n\omega\, d\omega}{[2(\cosh\vartheta - \cos\omega)]^{3/2}} = \coth\vartheta\frac{dg_n(\vartheta)}{d\vartheta} + \left(n^2 - \frac{1}{4}\right)g_n(\vartheta), \quad (23)$$

$$\alpha_n(\vartheta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin^2\omega\sinh^2\vartheta}{[2(\cosh\vartheta - \cos\omega)]^{7/2}}\cos n\omega\, d\omega = \frac{1}{15}\left[\frac{d^2\beta_n(\vartheta)}{d\vartheta^2} - \frac{d\beta_n(\vartheta)}{d\vartheta}\coth\vartheta\right].$$

The latter expressions for α_n and β_n are easily obtained by differentiating (14) with respect to ϑ, using Eq. (17).

3. GENERAL SOLUTION

In accordance with (10) – (12), the solution for ψ can be written in the form

$$\psi = [2(\cosh\vartheta - \cos\omega)]^{-1/2}\left[F_0(\vartheta) + 2\sum_{n=1}^{\infty}F_n(\vartheta)\cos n\omega\right]. \quad (25)$$

The plasma pressure vanishes at the surface of the toroid $\vartheta = \vartheta_0$ (or is constant); consequently the function $\psi(\vartheta_0, \omega)$ can be taken as a constant. We can set this constant equal to zero so that

$$\psi(\vartheta_0, \omega) = 0. \quad (26)$$

The expansion coefficients $F_n(\vartheta)$ which satisfy Eq. (26) for the inner and outer regions are

$$F_{in} = \frac{\gamma_n(\vartheta_0)}{g_n(\vartheta_0)} g_n(\vartheta) - \gamma_n(\vartheta), \quad (27)$$

$$F_{en} = C_n\left[g_n(\vartheta) - \frac{g_n(\vartheta_0)}{f_n(\vartheta_0)} f_n(\vartheta)\right]. \quad (28)$$

We assume that outside the volume current there is an azimuthal surface current J_S whose density is constant at equilibrium $j_S = \frac{\sinh\vartheta_0}{2\pi R_0} J_S$ = const; this azimuthal current component is equal to the current discontinuity in the tangential component of the magnetic field H_ω:

$$\frac{4\pi}{c} j_S = \frac{2J_S}{cR_0}\sinh\vartheta_0 = \{H_\omega\} = \frac{[2(\cosh\vartheta_0 - \cos\omega)]^{3/2}}{8\pi R_0^2 \sinh\vartheta_0}\left\{\frac{\partial F}{\partial\vartheta}\right\}, \quad (29)$$

where the symbol $\{\ldots\}$ denotes the discontinuity in the corresponding quantity at the point ϑ_0. We write this condition, which serves for determining the constant C_n, in another form:

$$\left\{\frac{\partial F}{\partial\vartheta}\right\} = 8\pi R_0^2 \frac{2J_S}{cR_0}\frac{\sinh^2\vartheta_0}{[2(\cosh\vartheta_0 - \cos\omega)]^{3/2}}$$

$$= \frac{16\pi R_0 J_S}{c}\left[\delta_0(\vartheta_0) + 2\sum_{n=1}^{\infty}\delta_n(\vartheta_0)\cos n\omega\right]. \quad (30)$$

Here

$$\delta_n(\vartheta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sinh^2\vartheta}{[2(\cosh\vartheta - \cos\omega)]^{3/2}}\cos n\omega\, d\omega$$

$$= -\left(n^2 - \frac{1}{4}\right) g_n(\vartheta) \quad (31)$$

[the integral is easily expressed in terms of $g_n(\vartheta)$ by means of (14) and (17)].

Substituting the derivatives of F_{in} and F_{en} in Eq. (30) we determine the coefficients C_n:

$$C_n = -\frac{\pi(n^2 - 1/4) f_n(\vartheta_0)}{\sinh\vartheta_0}\left\{\gamma_n'(\vartheta_0) - \gamma_n(\vartheta_0)\frac{g_n'(\vartheta_0)}{g_n(\vartheta_0)}\right.$$

$$\left. + \frac{16\pi R_0 J_S}{c}\left(n^2 - \frac{1}{4}\right) g_n(\vartheta_0)\right\}. \quad (32)$$

The general solution is given by Eqs. (25), (27), (28), and (32). The magnetic field outside the toroid is determined in terms of ψ_e by means of Eq. (7). The field, currents and pressure inside the toroid are determined in terms of ψ_i. In accordance with Eqs. (3), (8), (20), and (21), the pressure p and the function $I = crH_\varphi/2$ are given by the formulas

$$p = \frac{J_p}{2\pi^2 c R_0^3}\frac{\sinh^2\vartheta_0}{\cosh\vartheta_0}\psi_i(\vartheta, \omega) + \frac{J_S^2 + 2J_\varphi J_S}{2\pi c^2 R_0^2}\sinh^2\vartheta_0, \quad (33)$$

$$I^2 = c\frac{e^{\vartheta_0}\sinh\vartheta_0}{2\pi R_0}(J_\varphi - J_p)\psi_i(\vartheta,\omega) + I_e^2. \quad (34)$$

The second term in p takes account of the discontinuity in the pressure due to the surface current and I_e is the current which produces the external longitudinal field.

4. EXTERNAL FIELD

To determine the field, produced by the external conductors in order to contain the plasma in equilibrium, from the total field determined by the function $\psi_e(\vartheta, \omega)$ it is necessary to subtract off the self-field of the current which flows in the toroid. To find the self-field, once again it is necessary to solve (11), but this time with different boundary conditions. In particular, in place of the requirement that ψ and j_S be constant at the surface of the toroid we impose the condition that the field must vanish at infinity. This means that the solution in the external region is composed only of the functions $f_n(\vartheta)\cos n\omega$. We write the final result:

$$F_{ni}^{\text{self}} = \left\{\frac{\pi}{4\sinh\vartheta_0}(4n^2 - 1)\right.$$

$$\left.\times[\gamma_n'(\vartheta_0) f_n(\vartheta_0) - \gamma_n(\vartheta_0) f_n'(\vartheta_0)] + \frac{1}{c}LJ_S\right\}$$

$$\times g_n(\vartheta) - \gamma_n(\vartheta), \quad (35)$$

$$F_{ne}^{\text{self}} = \left\{\frac{\pi(4n^2 - 1)}{4\sinh\vartheta_0}\frac{g_n(\vartheta_0)}{f_n(\vartheta_0)}[\gamma_n'(\vartheta_0) f_n(\vartheta_0) - \gamma_n(\vartheta_0) f_n'(\vartheta_0)]\right.$$

$$\left. - \frac{\gamma_n(\vartheta_0)}{f_n(\vartheta_0)} + \frac{1}{c}LJ_S\frac{g_n(\vartheta_0)}{f_n(\vartheta_0)}\right\} f_n(\vartheta). \quad (36)$$

Here L is the self induction of a toroid with a strong skin effect:[2]

$$\frac{1}{L} = \frac{1}{2\pi^2 R_0}\left\{\frac{g_0(\vartheta_0)}{f_0(\vartheta_0)} - 2\sum_{n=1}^{\infty}\frac{1}{4n^2 - 1}\frac{g_n(\vartheta_0)}{f_n(\vartheta_0)}\right\}. \quad (37)$$

Subtracting (36) from (28) we find the coefficients $F_n(\vartheta)$ which determine the external field required for equilibrium. We write the values at infinity, where the functions $f_n(\vartheta)$ vanish:

$$F_{ne}^{\text{ext}} = -\left\{\frac{\pi^2 R_0 J_S}{c\sinh\vartheta_0}\left(n^2 - \frac{1}{4}\right)^2 g_n(\vartheta_0) f_n(\vartheta_0) + \frac{\pi(4n^2 - 1)}{4\sinh\vartheta_0}\right.$$

$$\left.\times\left[\gamma_n'(\vartheta_0) - \gamma_n(\vartheta_0)\frac{g_n'(\vartheta_0)}{g_n(\vartheta_0)}\right] f_n(\vartheta_0)\right\} g_n(\vartheta). \quad (38)$$

In principle the formulas which have been obtained allow us to compute the magnetic field configuration required for containing a plasma toroid of arbitrary cross section radius a in equilibrium. The corresponding equilibrium conditions for the limiting case $a/R_0 \to 0$ have been obtained by Osovets[3] who equilibrated the expansion forces of the circular current $(J_\varphi^2/2c^2)\partial L/\partial R$ to the contraction forces of the ring in the perpendicular magnetic field $2\pi RJH/c$.

5. CASE OF SMALL CURVATURE

In order to obtain a general idea of the nature of the solutions which have been obtained we consider the case of small curvature $a/R_0 \ll 1$, i.e., $e^{\vartheta_0} \gg 1$. A detailed examination of the functions $g_n(\vartheta)$ and $f_n(\vartheta)$ has been given by Fock.[2] In the limit, when $\vartheta \to \infty$ we find $(n \geq 2)$

$$g_0 = e^{\vartheta/2}, \quad g_1 = -\frac{1}{2} e^{-\vartheta/2}, \quad g_n = -\frac{(2n-3)!!}{n!\,2^n} e^{-(n-1/2)\vartheta},$$

$$f_0 = \frac{2}{\pi} e^{\vartheta/2}(\vartheta + 2\ln 2 - 2), \quad f_1 = \frac{2}{3\pi} e^{3\vartheta/2},$$

$$f_n = \frac{2^n}{\pi}\frac{(n-1)!}{(2n+1)!!} e^{(n+1/2)\vartheta}. \tag{39}$$

Correspondingly, from (22) – (24) we have

$$\alpha_0 = \frac{1}{8} e^{-3\vartheta/2}, \quad \beta_0 = \frac{1}{2} e^{-3\vartheta/2}, \quad \gamma_0 = \frac{2\pi R_0 J_\varphi}{c} e^{2\vartheta_0 - 3\vartheta/2},$$

$$\alpha_1 = \frac{7}{32} e^{-5\vartheta/2}, \quad \beta_1 = \frac{3}{8} e^{-5\vartheta/2},$$

$$\gamma_1 = \frac{2\pi R_0}{c}\left(J_p + \frac{3}{4} J_\varphi\right) e^{2\vartheta_0 - 5\vartheta/2}. \tag{40}$$

We write all solutions, limiting ourselves to two terms in the Fourier expansion for F.

a) Self-field of the current:

$$\psi_e^{\text{self}} = [2(\cosh\vartheta - \cos\omega)]^{-1}\{(2\pi^2 R_0/c)(J_\varphi + J_S) f_0(\vartheta)$$
$$+ (6\pi^2 R_0/c)\,e^{-2\vartheta_0}[(J_p + {}^3/_4 J_\varphi)$$
$$- (\vartheta_0 + 2\ln 2 - 2) J_S] f_1(\vartheta)\cos\omega\}. \tag{41}$$

When $\vartheta_0 = \infty$ only the first term remains; this represents the field of a linear circular current. When $e^{\vartheta} \ll 1$ we obtain the approximation formula

$$\psi_e^{\text{self}} = (4\pi R_0/c)(J_\varphi + J_S)(\vartheta + 2\ln 2 - 2)$$
$$+ (4\pi R_0/c)\{(J_\varphi + J_S)(\vartheta + 2\ln 2 - 2)e^{-\vartheta}$$
$$+ [J_p + {}^3/_4 J_\varphi - (\vartheta_0 + 2\ln 2 - 2) J_S] e^{-2\vartheta_0 + \vartheta}\}\cos\omega. \tag{42}$$

b) The equilibrium field (38) can be expressed conveniently in polar coordinates. In the approximation being used here $e^{-\vartheta}\cos\omega = (r - R_0)/2R_0$ and ψ^{ext} is of the form

$$\psi^{\text{ext}} = -(4\pi R_0/c)(J_S + J_\varphi)(\vartheta_0 + 2\ln 2 - 2)$$
$$- \{(2\pi/c)(J_S + J_\varphi)(\vartheta_0 + 2\ln 2 - 2)$$
$$+ (3\pi/c)[J_S + {}^2/_3(J_p + {}^3/_4 J_\varphi)]\}(r - R_0). \tag{43}$$

The magnetic field is determined by the formulas

$$H_r = -\frac{1}{2\pi r}\frac{\partial\psi}{\partial z} = 0,$$

$$H_z = \frac{1}{2\pi r}\frac{\partial\psi}{\partial r} = -\frac{J_S}{cR_0}\left(\vartheta_0 + 2\ln 2 - \frac{1}{2}\right)$$
$$- \frac{J_\varphi}{cR_0}\left(\vartheta_0 + 2\ln 2 - \frac{5}{4}\right) - \frac{J_p}{cR_0}, \tag{44}$$

where $\vartheta_0 = \ln(2R_0/a)$. When $J_\varphi = J_p = 0$, Eq. (44) coincides with the results obtained in references 1 and 4, and when $J_S = 0$, $J_\varphi = J_p$ (no longitudinal field), (44) coincides with the results obtained in reference 1.

c) We now find the equilibrium distributions of various quantities:

$$\psi_i = (2\pi R_0 J_\varphi/c)(1 - e^{2\vartheta_0 - 2\vartheta})$$
$$+ (4\pi R_0/c)(J_p + {}^5/_4 J_\varphi)e^{-\vartheta}(1 - e^{2\vartheta_0 - 2\vartheta})\cos\omega,$$

$$\psi_e \approx -(4\pi R_0/c)(J_\varphi + J_S)(\vartheta_0 - \vartheta)$$
$$- (4\pi R_0/c)e^{-\vartheta}\{(\vartheta_0 - \vartheta)(J_\varphi + J_S)$$
$$+ (J_p + {}^3/_4 J_\varphi + {}^3/_2 J_S)(1 - e^{2\vartheta - 2\vartheta_0})\}\cos\omega \tag{45}$$

The quantities p and I, given by (33) and (34), can be expressed conveniently in a local polar coordinate system ρ, ω, and φ, defined by the relations

$$r = R_0 + \rho\cos\omega, \quad z = \rho\sin\omega. \tag{46}$$

We write the values of p, I, H_ω in the absence of a surface current:

$$p = \frac{J_\varphi J_p}{\pi c^2 a^2}\left(1 - \frac{\rho^2}{a^2}\right) + \frac{J_p(J_p + {}^5/_4 J_\varphi)}{\pi c^2 a^2}\frac{a}{R_0}\frac{\rho}{a}\left(1 - \frac{\rho^2}{a^2}\right)\cos\omega,$$

$$I^2 - I_e^2 = 2\pi c^2 R_0^2 \frac{J_\varphi - J_p}{J_p} p(\rho, \omega),$$

$$H_\omega = -\frac{2J_\varphi}{ca}\frac{\rho}{a} + \frac{J_p}{cR_0}\left(1 - \frac{3\rho^2}{a^2}\right)\cos\omega$$
$$+ \frac{5}{4}\frac{J_\varphi}{cR_0}\left(1 + \frac{\rho^2}{5a^2}\right)\cos\omega. \tag{47}$$

1. If $J_\varphi J_p < 0$, this configuration is possible only in the presence of an external pressure p_e. We limit ourselves to the first term in p:

$$p = p_e - \frac{|J_\varphi J_p|}{\pi c^2 a^2}\left(1 - \frac{\rho^2}{a^2}\right). \tag{48}$$

Since $J_\varphi(J_\varphi - J_p) > 0$, the longitudinal magnetic field inside the toroid is larger than the applied field. A configuration of this kind can be in equilibrium without external fields.[1] It follows from Eq. (44) that this case is possible for a distributed current if $J_p = -J_\varphi[\ln(8R_0/a) - {}^5/_4]$.

In this case

$$I^2 = 2J_\varphi^2 \frac{R_0^2}{a^2}\left(\ln\frac{8R_0}{a} - \frac{1}{4}\right)\left(1 - \frac{\rho^2}{a^2}\right). \tag{49}$$

2. The case $J_{\varphi'} = 0$ is also only possible when $p_e \neq 0$ and, furthermore, if $I_e \neq 0$:

$$p = p_e + \frac{J_p^2}{\pi c^2 a^2}\frac{a}{R_0}\frac{\rho}{a}\left(1 - \frac{\rho^2}{a^2}\right)\cos\omega ,$$

$$I^2 = I_e^2 - 2J_p^2\frac{R_0}{a}\frac{\rho}{a}\left(1 - \frac{\rho^2}{a^2}\right)\cos\omega . \tag{50}$$

The pressure in the inner part of the toroid is smaller, and in the outer part is larger, than the external pressure. The currents in these two sections flows in opposite directions.

3. When $J_\varphi J_p > 0$, it is possible to have a configuration which is bounded by a vacuum. The internal longitudinal magnetic field can either be larger than the external field $I^2 > I_e^2$ if $|J_\varphi| > |J_p|$, or smaller than the external field ($|J_p| > |J_\varphi|$). The first case corresponds to a "paramagnetic" pinch and the second to a "diamagnetic" pinch. The distribution inside the pinch is given by (47).

4. When $J_p = 0$ we have the force-free field p = const. The internal longitudinal field is larger than the external field:

$$I^2 - I_e^2 = \frac{2R_0^2}{a^2}J_\varphi^2\left(1 - \frac{\rho^2}{a^2}\right) + \frac{5}{2}\frac{R_0}{a}J_\varphi^2\frac{\rho}{a}\left(1 - \frac{\rho^2}{a^2}\right)\cos\omega. \tag{51}$$

In conclusion we compare the distribution in the toroid with the distribution in a cylinder. The quantities j_φ, H_φ, ψ, and H_ω in the toroid correspond to j_z, H_z, A_z, and H_φ in a straight cylinder with axis along the z-axis. The equilibrium equation for a cylindrical pinch [$j_z H_\varphi = j_\varphi H_z - c\,dp/dr$,] taking account of the relations $H_\varphi = -dA_z/dr$, $j_\varphi = -(c/4\pi)\,dH_z/dr$, can be written in the form

$$j_z = c\,dp/dA_z + (c/8\pi)\,dH_z^2/dA_z. \tag{52}$$

This equation is the analog of (1) and (3). The analog of (4) is the equation $r^{-1}d(rH_\varphi)/dr = (4\pi/c)\,j_z$ or

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_z}{dr}\right) = -4\pi\frac{dp}{dA_z} - \frac{1}{2}\frac{dH_z^2}{dA_z}. \tag{53}$$

The conditions in (8), A = const and B = const, correspond to the relations $p = \mathrm{const}\cdot A_z$ and $H_z^2 = \mathrm{const}\cdot A_z$ in the case of the cylinder. In this case j_z = const and, consequently, p, A_z and H_z^2 have parabolic distributions.

APPENDIX

For the convenience of the reader we give the expansion of the functions $f_n(\vartheta)$ and $g_n(\vartheta)$ from Fock[2]

$$g_n(\vartheta) = \frac{\Gamma(n - 1/2)}{\Gamma(-1/2)\Gamma(n+1)} e^{-(n-1/2)\vartheta}(1 - e^{-2\vartheta})^2$$

$$\times F(n + 3/2,\ 3/2,\ n+1,\ e^{-2\vartheta}),$$

$$f_n(\vartheta) = \frac{1}{8}e^{-(n-1/2)\vartheta}(1 - e^{-2\vartheta})^2 F(n + 3/2,\ 3/2,\ 3,\ 1 - e^{-2\vartheta})$$

$$= \frac{2}{\pi}\frac{\Gamma(n - 1/2)}{\Gamma(-1/2)\Gamma(n+1)} e^{-(n-1/2)\vartheta}$$

$$\times(1 - e^{-2\vartheta})^2[-f_n^{(1)}(\vartheta) + f_n^{(2)}(\vartheta)].$$

Here F is the hypergeometric function:

$$F(n + 3/2,\ 3/2,\ n+1,\ e^{-2\vartheta}) = 1 + \frac{(2n+3)\,3}{(2n+2)\,2}e^{-2\vartheta}$$

$$+ \frac{(2n+3)(2n+5)\cdot 3\cdot 5}{(2n+2)(2n+4)\cdot 2\cdot 4}e^{-4\vartheta} + \ldots,$$

$f_n^{(1)}$ is the finite series

$$f_n^{(1)}(\vartheta) = \frac{2n}{2n+1}$$

$$\times\left\{e^{2\vartheta} + \frac{(2n-2)2}{(2n-1)1}e^{4\vartheta} + \frac{(2n-2)(2n-4)\cdot 2\cdot 4}{(2n-1)(2n-3)\cdot 1\cdot 3}e^{6\vartheta} + \ldots\right\},$$

$f_n^{(2)}$ is the infinite series

$$f_n^{(2)}(\vartheta) = \vartheta + 2\ln 2 - a_0 - a_n$$

$$+ \frac{(2n+3)3}{(2n+2)2}e^{-2\vartheta}(\vartheta + 2\ln 2 - a_1 - a_{n+1})$$

$$+ \frac{(2n+3)(2n+5)\cdot 3\cdot 5}{(2n+2)(2n+4)\cdot 2\cdot 4}$$

$$\times e^{-4\vartheta}(\vartheta + 2\ln 2 - a_2 - a_{n+2}) + \ldots,$$

where

$$a_0 = 1,\ a_s = a_{s-1} - \frac{1}{2s(2s+1)}$$

$$= 1 - \frac{1}{2} + \frac{1}{3}\ldots - \frac{1}{2s} + \frac{1}{2s+1}.$$

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