INCOMPATIBILITY OF THE CONDITIONS OF ANALYTICITY AND UNITARITY IN THE LEE MODEL

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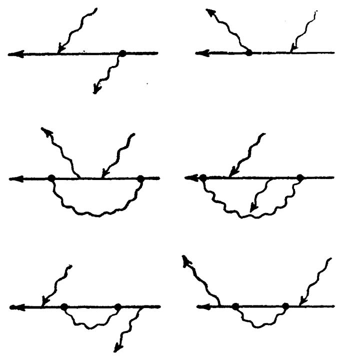
It is shown that outside the framework of the Hamiltonian formalism, and when one takes into account only the selection rules that are characteristic of the Lee model, the conditions of analyticity lead (even in the simplest $N + \theta$ sector) to a contradiction with the condition of unitarity. Owing to the existence of crossing symmetry this contradiction does not arise in the usual meson theories (at any rate for a static nucleon in the one-meson approximation, which in the Lee model is analogous to the case of the $N + \theta$ sector).

QUITE recently there have been[1,2] a number of interesting attempts to construct a theory of the strong interactions on the basis of a combination of the conditions of unitarity and analyticity. In this connection it is interesting to examine what such an approach gives in the simple case of the Lee model.[3] As is well known, in the usual Hamiltonian formalism of quantum field theory unphysical states appear in the Lee model,[4] and the theory is found to be self-contradictory.

It is shown below that outside the framework of the Hamiltonian formalism, and when we take into account only the selection rules that are characteristic of the Lee model (and which, as is well known, destroy the crossing symmetry of the theory), even in the simplest $N + \theta$ sector the condition of unitarity is incompatible with the condition of analyticity of the scattering amplitude. Thus, independently of the formalism used, just the assumption that the selection rule $V \rightleftarrows N + \theta$ holds, with the theory nonrelativistic in particles V and N, leads to clearly unphysical results. Unlike the Lee model, the usual meson theories, characterized by crossing symmetry, do not lead to a similar contradiction,* in any case for a static nucleon in the simplest one-meson approximation, which is analogous to the case of the $N + \theta$ sector in the Lee model.

Let us first examine in detail the case of meson theories that have crossing symmetry as a characteristic feature; this case has been studied in earlier papers.[5,6] For definiteness we shall speak of the theory of scalar charged mesons with scalar coupling and a static nucleon. In this case for $\mu/M \ll 1$, $\omega/M \ll 1$, where $\omega = (\mu^2 + k^2)^{1/2}$ is the energy of the meson, the meson-nucleon scattering amplitude depends only on ω. For a point

*A similar statement is contained in a paper by Mandelstam.[1]



a b

FIG. 1

interaction it is given by an analytic function $A(\omega)$, such that $A_+(\omega) = A(\omega + i\tau)$ for the scattering of π^+ mesons by protons and $A_-(\omega) = A(-\omega - i\tau)$ for the scattering of π^- [where $\tau \to 0$; these relations hold in the physical region $|\omega| \geq \mu$; in the region $|\omega| < \mu$ the function $A(\omega)$ is a real quantity]. The relation $A_-(\omega) = A_+^*(-\omega)$ expresses the crossing symmetry of the theory; it is not hard to see that this is a consequence of the fact that in the whole set of diagrams for the scattering of a meson by a nucleon each diagram, for example any of those of Fig. 1a, can be paired with another, as in Fig. 1b, in which the incident and scattered mesons are interchanged.

The function $A(\omega)$ satisfies the conditions of unitarity $(\omega > \mu)$

$$\operatorname{Im} A(\omega + i\tau) = \sqrt{\omega^2 - \mu^2}\, |A(\omega)|^2 + \ldots \qquad (1)$$

$$\operatorname{Im} A(-\omega - i\tau) = \sqrt{\omega^2 - \mu^2}\, |A(-\omega)|^2 + \ldots \qquad (2)$$

(where in the right members we have neglected terms corresponding to the occurrence of two or more mesons), and a condition of analyticity, which, on the assumption that for $\omega \to \infty$ the quantity $|A|^2$ falls off sufficiently rapidly, can be written in the form of the well known dispersion relation

$$A(\omega)=\frac{g^2}{\omega}+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\operatorname{Im}A(\omega'+i\tau)\,d\omega'}{\omega'-\omega}-\frac{1}{\pi}\int_{-\infty}^{-\mu}\frac{\operatorname{Im}A(\omega'-i\tau)\,d\omega'}{\omega'-\omega}\tag{3}$$

(here g is the coupling constant, normalized in a definite way).

As is well known, the appearance of the last term in the right member of Eq. (3) is due to the crossing symmetry, since because of the equation $A(\omega - i\tau) = A_-(-\omega)$ the function $A(\omega)$ has a branch point not only at $\omega = +\mu$ [where the amplitude $A_+(\omega)$ has a branch point] but also at $\omega = -\mu$, since here the amplitude $A_-(-\omega)$ has a branch point. Substitution of (1) and (2) in (3) leads to the Low equations, whose general solution has been found by Castillejo, Dalitz, and Dyson.[5] These authors remarked that the most general form of an analytic function satisfying the unitarity relations (1) and (2), having a pole with residue g^2 at $\omega = 0$, and having two branches for $|\omega| > \mu$, is

$$A(\omega)=-\frac{1}{\omega}\,\frac{1}{H(\omega)},\tag{4}$$

where

$$H(\omega)=-\frac{1}{g^2}+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\operatorname{Im}H(\omega'+i\tau)\,\omega d\omega'}{\omega'(\omega'-\omega)}-\frac{1}{\pi}\int_{-\infty}^{-\mu}\frac{\omega}{\omega'}\,\frac{\operatorname{Im}H(\omega'-i\tau)\,d\omega'}{\omega'-\omega}+R(\omega),\tag{5}$$

with

$$R(\omega)=\sum_n\frac{\omega}{\omega_n}\,\frac{R_n}{\omega_n-\omega}+\omega R_\infty.\tag{6}$$

Here R_n and R_∞ are real positive numbers, and the ω_n are also real quantities, namely the values of ω at which $A(\omega)$ is zero. It follows immediately from (1), (2), and (4) that

$$\operatorname{Im}H(\omega'\pm i\tau)=\pm\frac{\sqrt{\omega'^2-\mu^2}}{\omega'},\tag{7}$$

for $\omega' > \mu$ and $\omega' < -\mu$, respectively; therefore both the integrals in (5) [the integral $J_1(\omega)$ from μ to ∞ and the integral $J_2(\omega)$ from $-\infty$ to $-\mu$] can be calculated immediately:

$$J(\omega)=J_1(\omega)+J_2(\omega)=\frac{2\omega}{\pi}\int_{\mu}^{\infty}\frac{\sqrt{\omega'^2-\mu^2}\,d\omega'}{\omega'(\omega'^2-\omega^2-i\tau)}=\frac{\mu-\sqrt{\mu^2-\omega^2}}{\omega}\tag{8}$$

[for $\omega > \mu$, in the regions $\pm(\omega \pm i\tau)$ the root $(\mu^2 - \omega^2)^{1/2}$ is defined so that it takes the value $+i(\omega^2 - \mu^2)^{1/2}$]. Therefore, by (4), (5), and (8)

$$A(\omega)=\frac{g^2}{\omega}\{1-g^2[J(\omega)+R(\omega)]\}^{-1}.\tag{9}$$

This function will satisfy the relation (3) if the curly bracket in (9) does not vanish for any value of ω, i.e., if $A(\omega)$ has no other poles besides the pole at the point $\omega = 0$. It follows directly from (9), (8), and (6) that this requirement can be satisfied if all $|\omega_n| > \mu$ (except just one value $|\omega_{n_0}|$, which can be less than μ), and if

$$g^2<\frac{1}{1+R(\mu)}\tag{10}$$

[if $R(\omega) \equiv 0$, i.e., for that solution for which $\omega A(\omega)$ vanishes nowhere it is enough if $g^2 < 1$].

Thus from these results, obtained by Castillejo, Dalitz, and Dyson, it follows that in the case of the meson theories the unitarity and analyticity conditions (1) — (3) are compatible and determine the scattering amplitude in the form (9).

The situation is quite different in the case of the Lee model. Let us consider the simplest $N + \theta$ sector and denote by $a(\omega)$ the analytic function whose value at the point $\omega + i\tau$ $(\omega > \mu)$ determines the amplitude of the scattering $N + \theta$. Instead of Eqs. (1) — (3), the unitarity and analyticity conditions are now written in the form

$$\operatorname{Im}a(\omega+i\tau)=\sqrt{\omega^2-\mu^2}\,|a(\omega)|^2,\tag{11}$$

$$a(\omega)=\frac{g^2}{\epsilon_0-\omega}+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\operatorname{Im}a(\omega'+i\tau)}{\omega'-\omega}\,d\omega',\tag{12}$$

where $\epsilon_0 = M_V - M_N$, and from the condition of stability of the V particle it follows that $\epsilon_0 < \mu$. Unlike (1), Eq. (11) is exact, because transitions from the $N + \theta$ sector into other sectors are forbidden. There is no term in (12) analogous to the last term in (3), since now the values of the function $a(\omega)$ for negative ω (more exactly, for any $\omega < \mu$) do not have the physical meaning of the amplitude of some process, as they could in the meson theories.

Just as in the preceding case of the meson theories we can see that the most general form of the analytic function $a(\omega)$ that satisfies the unitarity condition (11), and that has a pole at $\omega = \epsilon_0$ and a branch point at $\omega = \mu$,* is

$$a(\omega)=-\frac{1}{\omega-\epsilon_0}\,\frac{1}{h(\omega)},\tag{13}$$

where

$$\operatorname{Im}h(\omega+i\tau)=\frac{\sqrt{\omega^2-\mu^2}}{\omega-\omega_0},\tag{14}$$

*The general form of the analytic function satisfied the condition, also satisfied by $A(\omega)$, that for complex ω the sign of $\operatorname{Im}\omega A(\omega)$ is the same as that of $\operatorname{Im}\omega$. This requirement follows from the analyticity condition (12) [and from Eq. (3) for $A(\omega)$].

$$h(\omega)=\frac{1}{g^2}+J_1(\omega)+\rho(\omega), \qquad (15)$$

$$J_1(\omega)=\frac{1}{\pi}\int_{\mu}^{\infty}\frac{(\omega-\varepsilon_0)}{(\omega'-\varepsilon_0)^2}\frac{\sqrt{\omega'^2-\mu^2}}{\omega'-\omega}d\omega', \qquad (16)$$

and, in analogy with Eq. (6),

$$\rho(\omega)=\sum_n\frac{\omega-\varepsilon_n}{\omega_n-\varepsilon_0}\frac{R_n}{\omega_n-\omega}+(\omega-\varepsilon_0)R_\infty. \qquad (17)$$

All the R_n and R_∞ are real and positive, and the ω_n are real.

The value of $a(\omega)$, that follows from (13) – (17)

$$a(\omega)=\frac{g^2}{\varepsilon_0-\omega}\{1+g^2[J_1(\omega)+\rho(\omega)]\}^{-1} \qquad (18)$$

coincides for $\rho(\omega)\equiv 0$ with the well known exact solution of the usual Hamiltonian equations of the Lee model in quantum field theory.

The analyticity relation (12) will be satisfied if $a(\omega)$ has no poles other than that at $\omega=\epsilon_0$. In particular, the curly brackets in (18) must not vanish for any real $\omega<\mu$ [in the case of the meson theories, owing to the crossing symmetry, it was enough to require that the curly brackets in (9) be nonvanishing only in the range $-\mu\leq\omega\leq\mu$; for $|\omega|>\mu$, in particular, and for $\omega<-\mu$, the integral $J(\omega)$ contained an imaginary part known to be different from zero]. This, however, can be so only for $g^2=0$. In other words, for any $g^2\neq 0$ ($g^2>0$) and for any choice of the numbers ω_n and R_n in (17), there is always a real value $\omega<\mu$ for which the curly brackets in (18) vanish.

What has been said follows immediately from (18) if we take into account the fact that, according to (16) and (17), $J_1'(\omega)>0$ and $\rho'(\omega)>0$, i.e., both these functions, $J_1(\omega)$ and $\rho(\omega)$, are increasing functions. Besides this it follows from (16) that at $\omega=\mu$ the integral $J_1(\omega)$ has a certain finite positive value, and for $\omega\to-\infty$ it increases without limit in absolute value, while remaining a negative quantity:

$$J_1(\omega)\approx-\frac{1}{\pi}\ln\frac{|\omega|}{\mu}, \quad \omega\to-\infty.$$

Therefore if we choose all the ω_n larger than μ the expression in curly brackets in (18) will be a a positive quantity at $\omega=\mu$ and a negative number of arbitrarily large absolute value for $\omega\to-\infty$. It is clear that at some value of ω this expression vanishes. If, on the other hand, we choose one (or several) ω_n smaller than μ and note that near such an ω_n the behavior of $\rho(\omega)$, by Eq. (17), is given by a discontinuous curve of the type shown in Fig. 2, it is all the more clear that at some value of ω in the region $\omega<\mu$ the curly brackets in (18) will have a (possibly multiple) zero.

Since (18) is the most general form of an analytic function that satisfies the condition of unitarity (11), it is clear that no solution of (11) and (12) exists. In other words, in the Lee model the conditions of unitarity and analyticity are not compatible.

In conclusion we shall demonstrate the incompatibility of Eqs. (11) and (12) by a different method. Let us set $\omega\to\omega+i\tau$ in Eq. (12) and write

$$a(\omega+i\tau)=(u(\omega)+iv(\omega))/2(\varepsilon_0-\omega-i\tau)$$

$$[v(\omega)\neq 0, \text{ for } \omega>\mu].$$

Then from Eq. (11) we have

$$v(\omega)=\frac{\varepsilon_0-\omega}{\sqrt{\omega^2-\mu^2}}\left\{1-\sqrt{1-\frac{\omega^2-\mu^2}{(\omega-\varepsilon_0)^2}u^2(\omega)}\right\}.$$

Substitution of this in Eq. (12) leads to an integral equation for the function $u(\omega)$:

$$u(\omega)=g^2+\frac{1}{\pi}\mathrm{P}$$

$$\times\int_{\mu}^{\infty}\frac{\omega-\varepsilon_0}{\omega-\omega'}\left\{1-\sqrt{1-\frac{\omega'^2-\mu^2}{(\omega'-\varepsilon_0)^2}u^2(\omega')}\right\}\frac{d\omega'}{\sqrt{\omega'^2-\mu^2}}$$

(the symbol P means that the integral is taken in the sense of the principal value). As can easily be seen, this equation cannot have any solution (real by definition) if $g^2\neq 0$. In fact, for $\omega\to\infty$ it gives

$$u(\infty)=g^2+\frac{1}{\pi}\int_{\mu}^{\infty}\left\{1-\sqrt{1-\frac{\omega'-\mu^2}{(\omega'-\varepsilon_0)^2}u^2(\omega')}\right\}\frac{d\omega'}{\sqrt{\omega'^2-\mu^2}}. \qquad (19)$$

Since the integral in the right member is positive, $u(\infty)$ cannot be equal to zero. But for $u(\infty)\neq 0$ the integral in the left member diverges logarithmically, and it then follows from the equation that $u(\infty)\to\infty$. On the other hand, $u(\infty)$ cannot exceed unity; otherwise, in the region $\omega'\to\infty$ the square root in the integrand becomes imaginary and the right side of (19) is a complex number.

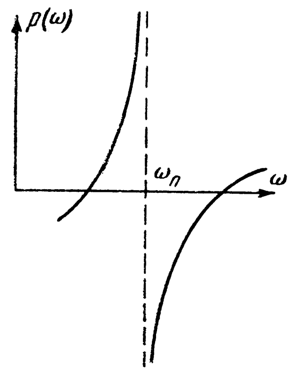


FIG. 2

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