ABSORPTION OF SOUND AND THE WIDTH OF SHOCK WAVES IN RELATIVISTIC HYDRODYNAMICS

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Submitted to JETP editor March 26, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 1000-1004 (October, 1959)

The absorption coefficient of sound due to viscosity and heat conduction is derived in relativistic hydrodynamics. The structure of relativistic low-intensity shock waves is considered.

INTRODUCTION

If a relativistic liquid possesses viscosity and heat conduction, then this leads to the gradual dissipation of the energy of the sound waves, i.e., to the absorption of the sound. The energy dissipated per unit time E_{mech} can be found by making use of the equation for the entropy increase, and also the expression for the maximum work performed in the transition from a given non-equilibrium state to a state of thermodynamic equilibrium (see reference 1).

The expression for the maximum work is

$$E_{\mathrm{mech}} = E_0 - E(S). \tag{1}$$

Here the energy $E(S) = \int \epsilon(s)\,dV_0$; integration is carried out over the volume of the liquid, dV_0 is the element of volume in the proper system of the observer; E_0 is the initial energy, and $E(S)$ is the energy of the body in the equilibrium state with the same entropy S which the body had initially. Starting out from (1), we can write the expression for the dissipated energy in the following form:

$$\dot{E}_{\mathrm{mech}} = -T_0 c \int \frac{dS}{ds}\,dV. \tag{2}$$

Here $T_0 = \partial\epsilon/\partial s$ is the temperature which the body would have had in the state of thermodynamic equilibrium; $ds = c\,d\tau$, where $d\tau$ is the differential of proper time; dV is the volume element in the laboratory system of the observer. The expression under the integral in Eq. (2) is determined by the equation for the growth of the entropy:[2]

$$\frac{dS}{ds} = \frac{S}{n}\frac{dn}{ds} + \frac{w}{T}\frac{\partial \nu^k}{\partial x^k} + \frac{1}{T}u^k\frac{\partial \tau_k^l}{\partial x^l}. \tag{3}$$

Here S is the entropy density, n is the density of number of particles per unit volume, while w is the heat function referred to a single particle; ν_i is the additional term in the four vector of the density of material flow n_i and τ_{ik} is the four-tensor of the viscosity (see reference 2).

CALCULATION OF THE SOUND ABSORPTION COEFFICIENT

For calculation of the dissipation of energy in the sound wave we make use of the fact that the velocity of motion of the particles of the liquid v in the sound wave is small, and that the motion takes place adiabatically. Then, taking it into account that the temperature at an infinitely distant surface tends to a constant limit, we obtain

$$E_{\mathrm{mech}} \approx -T_0 \int \frac{s}{n}\left(v\frac{\partial n}{\partial x} + \frac{\partial n}{\partial t}\right) dV + T_0 \varkappa \int \Big[-\frac{2}{w^2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{2}{wT}\frac{\partial T}{\partial x}\frac{\partial w}{\partial x} + \frac{1}{w}\frac{\partial^2 w}{\partial x^2} - \frac{1}{T^2}\left(\frac{\partial T}{\partial x}\right)^2 \Big] dV - T_0 \frac{4\eta/3 + \zeta}{c} \int \frac{1}{T}\left(\frac{\partial v}{\partial x}\right)^2 dV. \tag{4}$$

(Here η and ζ are the two coefficients of viscosity, while κ is the coefficient of heat conduction, taken in correspondence with its nonrelativistic value.)

For the determination of the integrals entering into (4), we make use of a series of relations which hold for a sound wave and also of a number of thermodynamic relations between arbitrary thermodynamical quantities. It is not difficult to show (see reference 2) that the following relations hold:

$$p' = (\partial p / \partial \varepsilon)_\sigma \varepsilon' = (W / c^2) c_0 v,$$
$$w' = (\partial w / \partial p)_\sigma p' = (W / nc^2) c_0 v,$$
$$T' = (\partial T / \partial p)_\sigma p' = (W / c^2) c_0 (\partial T / \partial p)_\sigma v \tag{5}$$

(c_0 is the sound velocity, W is the heat function per unit volume). The primed quantities in these relations refer to small increments in the sound wave, the derivatives are taken at constant entropy σ per single particle, and the velocity $v = \partial\varphi/\partial x$, where φ is the velocity potential. The mean energy density of the sound wave in relativistic hydrodynamics in the laboratory system is given by the expression

$$\bar{v}^2 (\varepsilon + 2p) / c^2. \quad (6)$$

If we denote by

$$c_p = T (\partial\sigma / \partial T)_p, \qquad c_V = T (\partial\sigma / \partial T)_V \quad (6a)$$

the heat capacity per single particle at constant pressure and constant volume $1/n = mV$, respectively, then, if we make use of the thermodynamical identities

$$d\varepsilon = nTd\sigma - n^2 w d(1/n), \qquad dw = Td\sigma + (1/n)\,dp,$$
$$d\mu = -\sigma dT + (1/n)\,dp, \quad (7)$$

we can obtain the following formulas:

$$c_V - c_p = T\left[\frac{\partial}{\partial T}\left(\frac{1}{n}\right)\right]_p^2 \Big/ \frac{\partial}{\partial p}\left(\frac{1}{n}\right)_T, \quad (8)$$

$$\left(\frac{\partial T}{\partial p}\right)_\sigma = \frac{T}{c_p}\frac{\partial}{\partial T}\left(\frac{1}{n}\right)_p, \quad (9)$$

$$\left[\frac{\partial(1/n)}{\partial p}\right]_\sigma = \frac{c_V}{c_p}\left[\frac{\partial(1/n)}{\partial p}\right]_T. \quad (10)$$

Making use of (9), and the value of the derivatives

$$\frac{\partial p}{\partial(1/n)_\sigma} = -Wnc_0^2/c^2, \quad (10a)$$

we write out Eq. (8) in the form

$$c_V - c_p = -\frac{c_V T W c_0^2}{c_p n c^2}\left[n\frac{\partial(1/n)}{\partial T}\right]_p^2. \quad (11)$$

The integrals entering into (4) will be computed for a plane wave of the form $v = v_0 \cos(kx - \omega t)$, the time average energy of such a wave in a volume V_0 of liquid being, in accord with (6),

$$\bar{E} = \tfrac{1}{2} v_0^2 V_0 (\varepsilon + 2p)/c^2. \quad (12)$$

The first integral in Eq. (4) vanishes in the mean; the same also applies to the integral of $(\partial^2 w/\partial x^2)/w$. Using (8) — (12) and

$$\left(\frac{\partial T}{\partial p}\right)_\sigma = \frac{T\beta}{c_p n}, \qquad \beta = n\frac{\partial}{\partial T}\left(\frac{1}{n}\right)_p \quad (12a)$$

(β is the coefficient of thermal expansion), the sound absorption coefficient

$$\gamma = |\bar{E}_{\text{mech}}| / 2c_0\bar{E} \quad (13)$$

takes the following final form in relativistic hydrodynamics:

$$\gamma = \frac{\omega^2 c^2}{2(\varepsilon + 2p)c_0^2}\left[\frac{1}{c}\left(\frac{4}{3}\eta + \zeta\right) + \varkappa\left(\frac{W}{nc^2}\right)\left(\frac{1}{c_V} - \frac{1}{c_p}\right) - \frac{2\varkappa}{c^2}\left(\frac{1}{c_V} - \frac{1}{c_p}\right)\frac{c_p}{\beta}\left(1 - \frac{c_p}{\beta w}\right)\right] \quad (14)$$

(where $k = \omega/c_0$). In the nonrelativistic limit, (14) goes over into the usual expression for the sound absorption coefficient.

THICKNESS OF SHOCK WAVES

It is well known that a weak shock wave in nonrelativistic hydrodynamics has a finite thickness which is inversely proportional to the amplitude of the wave.

The distribution of thermodynamical quantities over the thickness of the shock wave is found with the help of the usual hydrodynamical laws of conservation of mass, energy and momentum, with account of streaming produced by viscosity and heat conduction. In relativistic hydrodynamics, the corresponding conservation laws have the following form:

$$\frac{\bar{n}\bar{v}}{(1 - \bar{v}^2/c^2)^{1/2}} - \frac{\varkappa}{1 - \bar{v}^2/c^2}\left(\frac{T}{w}\right)^2\left[\frac{\partial}{\partial x}\left(\frac{\bar{\mu}}{T}\right) + \frac{\bar{v}}{c^2}\frac{\partial}{\partial t}\left(\frac{\bar{\mu}}{T}\right)\right] = \frac{nv}{\sqrt{1 - v^2/c^2}} = j, \quad (15)$$

$$\bar{n}\bar{w}\frac{\bar{v}^2}{c^2(1 - \bar{v}^2/c^2)} + \bar{p} - \left(\frac{4}{3}\eta + \zeta\right)\frac{1}{c(1 - \bar{v}^2/c^2)^{3/2}}\left[\frac{\partial\bar{v}}{\partial x} + \frac{\bar{v}}{c^2}\frac{\partial\bar{v}}{\partial t}\right] = nw\frac{v^2}{c^2(1 - v^2/c^2)} + p, \quad (16)$$

$$\bar{n}\bar{w}\frac{\bar{v}}{c(1 - \bar{v}^2/c^2)} - \left(\frac{4}{3}\eta + \zeta\right)\frac{\bar{v}}{c^2(1 - \bar{v}^2/c^2)^{3/2}}\left[\frac{\partial\bar{v}}{\partial x} + \frac{\bar{v}}{c^2}\frac{\partial\bar{v}}{\partial t}\right] = nw\frac{v}{c^2(1 - v^2/c^2)} \quad (17)$$

(the shock wave moves from the right to the left, the state in front of the shock wave is denoted without bars).

Proceeding in the usual fashion, we expand the values of all quantities on the shock wave in a series of powers of the entropy jump $\Delta\sigma = \bar{\sigma} - \sigma$ and the pressure jump $\Delta p = \bar{p} - p$. Taking the thermodynamic identities (7) into account, we have

$$\bar{w} - w = T\Delta\sigma + \frac{1}{n}\Delta p + \frac{1}{2}\frac{\partial}{\partial p}\left(\frac{1}{n}\right)_\sigma \Delta p^2,$$

$$\bar{n} - n = -n^2\frac{\partial}{\partial\sigma}\left(\frac{1}{n}\right)_p \Delta\sigma - n^2\frac{\partial}{\partial p}\left(\frac{1}{n}\right)_\sigma \times \Delta p - n^2\left[\frac{1}{2}\frac{\partial^2}{\partial p^2}\left(\frac{1}{n}\right)_\sigma - n\left(\frac{\partial}{\partial p}\left(\frac{1}{n}\right)_\sigma\right)^2\right]\Delta p^2. \quad (18)$$

Here we have neglected terms in $\Delta\sigma$ and Δp higher than the first and third, because $\Delta\sigma$ (as we shall see below) has an order of smallness not higher than second. Further, we express the derivatives in (15) in terms of derivatives of p and σ. In this case, it is necessary to recall that differentiation with respect to x and t increases the order of smallness of quantities per unit value (since the width of the shock wave is inversely proportional to the amplitude of the wave). Therefore, the derivatives $\partial p/\partial x$ and $\partial p/\partial t$ are quantities of second order of smallness, while the derivatives $\partial\sigma/\partial x$ and $\partial\sigma/\partial t$ are third order. Thus, on the whole, the mass flow brought about by the thermal conductivity has a second order of smallness in Eq. (15). As a result we obtain (15) in the form

$$\frac{\overline{n}\,\overline{v}}{(1-\overline{v}^2/c^2)^{1/2}} = \varkappa\left(\frac{T}{w}\right)^2\frac{1}{T}\left(\frac{\partial w}{\partial p_\sigma} - \frac{w}{T}\frac{\partial T}{\partial p_\sigma}\right)\left[\left(1+\frac{j^2}{n^2c^2}\right)\frac{\partial p}{\partial x} + \frac{j}{nc^2}\left(1+\frac{j^2}{n^2c^2}\right)^{1/2}\frac{\partial p}{\partial t}\right] + j. \tag{19}$$

It is then easy to find the expansion for $\overline{v}$. Hence, substituting the resultant expansions in (16), we find an equation which connects the pressure and entropy jumps (this relation is too cumbersome to write down here).

To determine the energy jump $\Delta\sigma$ on the discontinuity, it is necessary to make use of both Eqs. (16) and (17), first subtracting the second, multiplied by $\overline{v}/c$, from the first. Carrying out a number of transformations of the thermodynamical derivatives, we finally obtain the following expression of second order relative to Δp:

$$\left[\left(1+\frac{j^2}{n^2c^2}\right) + \frac{j^2}{n^2c^2}wn^2\frac{\partial}{\partial p}\left(\frac{1}{n}\right)\right]\Delta p + \left[\frac{3}{2}\frac{j^2}{n^2c^2}n\frac{\partial}{\partial p}\left(\frac{1}{n}\right) + \frac{1}{2}\frac{j^2}{n^2c^2}wn^2\frac{\partial^2}{\partial p^2}\left(\frac{1}{n}\right)\right]\Delta p^2$$
$$= j\left(1+\frac{j^2}{n^2c^2}\right)\frac{\partial}{\partial p}\left(\frac{1}{n}\right)_\sigma\left[\frac{\partial p}{\partial x} + \frac{j}{nc^2(1+j^2/n^2c^2)^{1/2}}\frac{\partial p}{\partial t}\right]a_1, \tag{20}$$

where

$$a = \frac{1}{c}\left(\frac{4}{3}\eta + \zeta\right) + \varkappa\left(\frac{W}{nc^2}\right)\left(\frac{1}{c_V} - \frac{1}{c_p}\right) - \frac{2\varkappa}{c^2}\left(\frac{1}{c_V} - \frac{1}{c_p}\right)\frac{c_p}{\beta}\left(1 - \frac{c_p}{\beta w}\right). \tag{20a}$$

The right side of Eq. (20) vanishes along with $\partial\overline{p}/\partial x$ and $\partial\overline{p}/\partial t$ at great distances on both sides of the surface of discontinuity (a replacement of $\partial p/\partial x$ by $\partial\overline{p}/\partial x$ is valid with accuracy up to terms of third order of smallness). At these distances, the pressure is equal to p and p_1, respectively. In other words, the quadratic (in p) expression on the left side of (20) has the roots $\overline{p} = p$ and $\overline{p} = p_1$. Therefore Eq. (20) can be represented in the form

$$\left[\frac{3}{2}\frac{j^2}{n^2c^2}n\frac{\partial}{\partial p}\left(\frac{1}{n}\right) + \frac{1}{2}\frac{j^2}{n^2c^2}wn^2\frac{\partial^2}{\partial p^2}\left(\frac{1}{n}\right)\right](\overline{p}-p)(\overline{p}-p_1)$$
$$= a_1 j\left(1+\frac{j^2}{n^2c^2}\right)\frac{\partial}{\partial p}\left(\frac{1}{n}\right)_\sigma\left[\frac{\partial\overline{p}}{\partial x} + \frac{j}{nc^2(1+j^2/n^2c^2)^{1/2}}\frac{\partial\overline{p}}{\partial t}\right]. \tag{21}$$

From the system of ordinary differential equations which correspond to Eq. (21), we find, by integrating with respect to x,

$$\overline{p} - \frac{p_1+p}{2} = \frac{p_1-p}{2}\tanh\frac{x}{\delta_x}, \tag{22}$$

where

$$\delta_x = \frac{8a(\varepsilon+2p)/n^2W}{(p_1-p)\left[\frac{3}{nc^2}\frac{\partial}{\partial p}\left(\frac{1}{n}\right) + \left(\frac{W}{nc^2}\right)\frac{\partial^2}{\partial p^2}\left(\frac{1}{n}\right)\right]\left(1-\frac{c_0^2}{c^2}\right)^{1/2}}. \tag{23}$$

Here it is taken into account that the velocity of the weak shock wave in zeroth approximation is equal to the sound velocity c_0^3 and therefore can be expanded to the same approximation:

$$j/n = c_0/\sqrt{1-c_0^2/c^2}. \tag{23a}$$

As expected, the quantity a is related to the sound absorption coefficient (14), namely $\gamma = a\omega^2$.

The quantity δ_x determines the width of the shock wave in relativistic hydrodynamics. We see that even here the width of the shock wave is inversely proportional to the amplitude of the wave. In the nonrelativistic limit, the first term in the square brackets in the denominator can be neglected in comparison with the second; then, (23) reduces to the well-known nonrelativistic expression.

Integration over t gives

$$\overline{p} - \frac{p_1+p}{2} = \frac{p_1-p}{2}\tanh\frac{t}{\delta_t}, \tag{24}$$

where

$$\delta_t = \frac{8a(\varepsilon+2p)c_0/n^2Wc^2}{(p_1-p)\left[\frac{3}{nc^2}\frac{\partial}{\partial p}\left(\frac{1}{n}\right) + \left(\frac{W}{nc^2}\right)\frac{\partial^2}{\partial p^2}\left(\frac{1}{n}\right)\right]\left(1-\frac{c_0^2}{c^2}\right)^{1/2}}; \tag{25}$$

δ_t determines the variation of the pressure over the thickness of the shock wave as a function of time. In the nonrelativistic limit, δ_t tends to zero.

For the variation of the entropy inside the discontinuity, we have the following results:

$$\Delta\sigma = \frac{\varkappa n}{16c_0aT}\left(\frac{\partial T}{\partial p}\right)_\sigma\left[\frac{3}{nc^2}\frac{\partial}{\partial p}\left(\frac{1}{n}\right) + \left(\frac{W}{nc^2}\right)\frac{\partial^2}{\partial p^2}\left(\frac{1}{n}\right)\right](p_1-p)^2 \times\left(\cosh^{-2}\frac{x}{\delta_x} + \cosh^{-2}\frac{t}{\delta_t}\right)\left(1 - \frac{T}{nw\partial T/\partial p_\sigma}\right). \tag{26}$$

The entropy reaches a maximum inside the discontinuity (for x = 0 and t = 0). At large distances on both sides of the shock wave, for $x \to \pm\infty$ and $t \to \pm\infty$, this formula gives $\overline{\sigma} = \sigma$. This is connected with the fact that in relativistic hydrodynamics, as well as in nonrelativistic, the total discontinuity in the entropy is a quantity of third order in Δp, while $\overline{\sigma} - \sigma$ is of second order.

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Translated by R. T. Beyer
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