SIMPLE WAVES IN THE CHEW-GOLDBERGER-LOW APPROXIMATION

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Simple waves in a plasma with anisotropic pressure are treated in the Chew-Goldberger-Low approximation. It is shown that there exist three types of simple waves. The direction of the variations of the magnetohydrodynamic quantities in these waves is investigated.

CHEW, Goldberger, and Low[1] have shown that a rarefied plasma in a magnetic field, in which collisions play a negligible role, can be described by a system of magnetohydrodynamic equations with an anisotropic pressure. Small vibrations of the plasma have been studied by means of these equations.[2] It is of interest to use these same equations for the study of nonlinear motions of the plasma, and primarily for the study of simple waves. The present paper is devoted to this last problem.

1. In the Chew-Goldberger-Low approximation the system of magnetohydrodynamic equations has the following form:

$$\rho\frac{d\mathbf{v}}{dt} = \mathbf{F} + \frac{1}{4\pi}(\operatorname{curl}\mathbf{H})\times\mathbf{H}, \qquad F_i = -\frac{\partial p_{ik}}{\partial x_k},$$

$$\frac{\partial\mathbf{H}}{dt} = \operatorname{curl}[\mathbf{v}\times\mathbf{H}], \qquad \operatorname{div}\mathbf{H} = 0, \qquad \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0.$$

$$p_{ik} = p_\perp\delta_{ik} + (p_\parallel - p_\perp)h_ih_k, \qquad \mathbf{h} = \mathbf{H}/H,$$

$$\frac{d}{dt}\left(\frac{p_\perp}{\rho H}\right) = 0, \qquad \frac{d}{dt}\left(\frac{p_\parallel H^2}{\rho^3}\right) = 0. \tag{1}$$

We shall consider one-dimensional simple waves, in which all the magnetohydrodynamic quantities are functions of one such quantity, for example ρ, which in turn depends on the coordinate x and the time t:

$$x - V_{\mathrm{ph}}(\rho)\,t = f(\rho), \tag{2}$$

where $V_{\mathrm{ph}}(\rho)$ is the rate of displacement of a point at which the density ρ has a given value, and $f(\rho)$ is the function inverse to the density distribution $\rho(x)$ at the initial time $t = 0$.

In regions of compression $f'(\rho) < 0$; in regions of rarefaction $f'(\rho) > 0$; and the condition for self-similar waves is $f(\rho) \equiv 0$.

Simple waves are closely related to waves of small amplitude.[3] The differential equations connecting the changes of all the magnetohydrodynamic quantities with the change of the density ρ are easily found from the relations between the amplitudes of the waves of small intensity.

Just as in magnetohydrodynamics with a scalar pressure, there exist three types of simple waves, in which the magnetohydrodynamic quantities satisfy the following differential equations.

Alfven waves:*

$$dv_z = -(u_A/H_x)\,dH_z, \qquad dv_y = -(u_A/H_x)\,dH_y,$$

$$H_ydH_y + H_zdH_z = 0,$$

$$dv_x = dH_x = d\rho = dp_\parallel = dp_\perp = 0, \tag{3}$$

where u_A is the speed of the Alfven waves, defined by the formula

$$u_A = V_{Ax}\sqrt{1 + 4\pi H^{-2}(p_\perp - p_\parallel)}, \qquad \mathbf{V}_A = \mathbf{H}/\sqrt{4\pi\rho}. \tag{4}$$

Magnetoacoustic waves

$$\frac{dv_x}{d\rho} = \frac{u_\pm}{\rho}, \quad \frac{dH_x}{d\rho} = 0, \quad \frac{dv_z}{d\rho} = 0, \quad \frac{dH_z}{d\rho} = 0,$$

$$\frac{dH_y}{d\rho} = \frac{H^2}{\rho H_y}\left(1 + \frac{s_\perp^2 H_x^2}{u_\pm^2H^2 - 3s_\parallel^2H_x^2}\right),$$

$$\frac{dv_y}{d\rho} = -\frac{u_\pm}{\rho}\frac{H_x}{H_y}\left(1 - \frac{s_\perp^2 H_y^2}{u_\pm^2H^2 - 3s_\parallel^2H_x^2}\right)\left(1 + \frac{s_\perp^2 H_x^2}{u_\pm^2H^2 - 3s_\parallel^2H_x^2}\right)^{-1},$$

$$\frac{dp_\parallel}{d\rho} = \frac{p_\parallel}{\rho}\left[1 + \frac{2s_\perp^2H_x^2}{u_\pm^2H^2 + H_x^2(s_\perp^2 - 3s_\parallel^2)}\right],$$

$$\frac{dp_\perp}{d\rho} = \frac{p_\perp}{\rho}\left[1 + \frac{u_\pm^2H^2 - 3H_x^2s_\parallel^2}{u_\pm^2H^2 + H_x^2(s_\perp^2 - 3s_\parallel^2)}\right], \tag{5}$$

where

$$s_\parallel^2 = p_\parallel/\rho, \quad s_\perp^2 = p_\perp/\rho.$$

The speeds $u_\pm$ of the magnetoacoustic waves are given by the formula

*In the limiting case $p \ll H$ these waves have been discussed by Ferraro.[4]

$$u_\pm = \left\{\frac{1}{2}[s_\perp^2(2-\cos^2\varphi)+2s_\parallel^2\cos^2\varphi+V_A^2]\right.$$
$$\pm\left[\frac{1}{4}(s_\perp^2(2-\cos^2\varphi)+2s_\parallel^2\cos^2\varphi+V_A^2)^2\right.$$
$$+s_\perp^4\sin^2\varphi\cos^2\varphi-3s_\parallel^2s_\perp^2\cos^2\varphi(2-\cos^2\varphi)$$
$$\left.\left.+3s_\parallel^4\cos^4\varphi-3s_\parallel^2V_A^2\cos^2\varphi\right]^{1/2}\right\}^{1/2}. \quad (6)$$

The plus sign corresponds to the fast, the minus to the slow, acoustic wave; φ is the angle between the direction of the magnetic field and the direction of propagation of the wave.

The quantity V_{ph} appearing in (2) is connected with the phase velocities in the following ways:

$$V_{ph}=v_x+u_A \quad (7)$$

for the Alfven waves, and

$$V_{ph}=v_x+u_\pm$$

for the magnetoacoustic waves.

It follows from (3) that $dV_{ph}/d\rho = 0$; this means that the Alfven wave is propagated without change of shape.

2. The study of (5) in general form entails great mathematical difficulties. We shall confine ourselves to the most interesting case, in which the hydrostatic pressure is much smaller than the magnetic pressure:

$$s_\perp \ll V_A, \quad s_\parallel \ll V_A.$$

Under these conditions the formulas (5) and (6) take the following form for the fast magnetoacoustic wave:

$$dv_x/d\rho = u_+/\rho, \quad dv_y/d\rho = -u_+H_x/\rho H_y,$$
$$dH_y/d\rho = H^2/\rho H_y,$$
$$dp_\parallel/d\rho = p_\parallel/\rho, \quad dp_\perp/d\rho = 2p_\perp/\rho, \quad u_+ = V_A. \quad (9)$$

From Eq. (9) we have*

$$dV_{ph}/d\rho > 0. \quad (10)$$

This means that in regions of rarefaction the density gradient decreases, and that it increases in regions of compression.

From the relation (2) and the equation of continuity it follows that

$$-\frac{1}{u_\pm(\rho)}[tV'_{ph}(\rho)+f'(\rho)]\frac{d\rho}{dt}=1. \quad (11)$$

*This fact has been noted by Sagdeev[5] in the case in which the direction of propagation of the wave is perpendicular to the magnetic field.

It can be seen from this that in regions of rarefaction ($f' > 0$) and in self-similar waves ($f \equiv 0$) the density decreases. In regions of compression ($f' < 0$) the density increases up to the point at which the expression in square brackets ceases to be negative. A compressional shock wave appears when this expression becomes zero.

It follows from the relations (9) that in a fast magnetoacoustic wave the quantities $p_\parallel$, $p_\perp$, H, $p_\perp/p_\parallel$ change in the same direction as the density.*

3. Let us now consider the slow magnetoacoustic wave. With the conditions $s_\perp \ll V_A$, $s_\parallel \ll V_A$ satisfied, the equations that hold on a slow magnetoacoustic wave are

$$\frac{dv_x}{d\rho}=\frac{u_-}{\rho}, \quad \frac{dv_y}{d\rho}=\frac{H_y}{H_x}\frac{u_-}{\rho}, \quad \frac{dp_\parallel}{d\rho}=\frac{3p_\parallel}{\rho}, \quad \frac{dp_\perp}{d\rho}=\frac{p_\perp}{\rho},$$
$$H_y\frac{dH_y}{d\rho}=\frac{s_\perp^2V_A^2H^4}{-\rho A(\rho)}, \quad 2u_-\frac{dV_\Phi}{d\rho}=\frac{6s_\parallel^2s_\perp^2V_A^2H_x^2}{\rho A(\rho)},$$
$$u_-=s_\parallel\sqrt{3}\cos\varphi, \quad A(\rho)=s_\perp^4H_y^2-3s_\parallel^2H_x^2(s_\perp^2-s_\parallel^2). \quad (12)$$

We must distinguish two cases, depending on the sign of the quantity $A(\rho)$.

1) Normal case†

$$A(\rho)>0, \quad dV_{ph}/d\rho>0.$$

Here the density changes in just the same way as in the fast magnetoacoustic wave. In particular, shock waves are formed in regions of compression, and the self-similar waves are waves of rarefaction. According to Eq. (12) the changes of the quantities $p_\parallel$, $p_\perp$ are in the same direction as those of the density, and those of the quantities H and $p_\perp/p_\parallel$ are in the opposite direction.

2) Anomalous case‡

$$A(\rho)<0, \quad dV_{ph}/d\rho<0.$$

Here the density gradient decreases in regions of compression, and increases in regions of rarefaction. It follows from (11) that in regions of compression the density rises, and in regions of rarefaction it falls. In self-similar waves the density rises. According to Eq. (12), in the anomalous case the quantities $p_\parallel$, $p_\perp$, and H change in the same direction as the density, and the quantity $p_\perp/p_\parallel$

*The decrease of the ratio $p_\perp/p_\parallel$ with decrease of the magnetic field is due to the conservation of a particle's magnetic moment.[6]

†This case is always realized, for example, if the wave is propagated perpendicular to the magnetic field ($H_x = 0$), or if the plasma is isotropic ($p_\parallel = p_\perp$).

‡This case is realized, for example, if the wave is propagated along the magnetic field ($H_y = 0$) and the quantity $p_\perp/p_\parallel$ is greater than unity.

changes in the opposite direction.*

It follows from (11) that shock waves arise in regions of rarefaction. Thus in this case, unlike that of magnetohydrodynamics with a scalar pressure, shock waves of rarefaction can be produced.

At the instant of production of a shock wave the plasma has the density ρ_s that satisfies the equation[8]

$$f''(\rho_s)/f'(\rho_s) = V''_{ph}(\rho_s)/V'_{ph}(\rho_s). \quad (13)$$

We note that in both the normal case and the anomalous case the quantity A increases: $dA/dt > 0$. Therefore the plasma can be in the anomalous state for only a finite time. The density value ρ_n at which the transition from the anomalous to the normal state occurs satisfies the equation $A(\rho_n) = 0$.

If $\rho_s > \rho_n$, a shock wave is formed in the region of rarefaction; if $\rho_n > \rho_s$, the plasma makes the transition to the normal state before a shock wave of rarefaction is formed. Which event happens first depends on the initial density distribution $f(\rho)$.

We note that the point ρ_n is not a singular point for the system of differential equations (12). At this point we merely have vanishing of the derivatives $d\rho/dH_y$, $dp_{\parallel}/dH_y$, $dp_{\perp}/dH_y$, dv_x/dH_y, and dv_y/dH_y.

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*This is in agreement with Parker's[7] result of an inverse dependence between $p_{\perp}/p_{\parallel}$ and H in the static case when the particles move along the magnetic field. The decrease of $p_{\perp}/p_{\parallel}$ with the increase of H, in spite of the conservation of magnetic moment (proportional to $p_{\perp}/H$) is explained by the fact that particles with larger values of $p_{\perp}/p_{\parallel}$ are more strongly reflected from the region of strong magnetic field.[6]

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