RELATIVISTIC TRANSPORT EQUATIONS FOR A PLASMA. I.

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A connection is established between the definitions of the probability of a state and the distribution functions given in the papers of various authors, based on the example of a transport equation for a charged particle in an external electromagnetic field.

A random function which determines the number of particles in a volume element in phase space is introduced. The electromagnetic field strengths or the numbers of oscillators are also considered as random functions. The set of equations for these functions serves as the basis for deriving a chain of equations connecting the moments of the random functions or the corresponding distribution functions of different orders. Through an approximation to this chain of equations we obtain a set of relativistic self-consistent equations. We give relativistic expressions for the dispersion equations for the transverse and longitudinal plasma waves. A variational principle for a relativistic plasma is considered.

THE possibility of a consistent derivation of the relativistic transport equations for a plasma is of interest in connection with the increasing importance of the theory of the high-temperature plasma.

The most general method to obtain approximate transport equations in the non-relativistic approximation is Bogolyubov's method[1] which is based upon an approximate solution of a chain of equations for distribution functions.

One must also construct a chain of relativistic equations for distribution functions to derive the transport equations in the relativistic case. Different forms of approximating higher distribution functions in terms of lower ones in such a chain makes it possible to obtain the appropriate transport equations: the equation with a self-consistent field, the Fokker-Planck equation for a relativistic plasma, the transport equation taking radiation into account, and so on. The present paper is also devoted to this problem.

The number of papers in which the relativistic transport equations for a plasma have been considered is very small.[2-4]

In the papers by the author[2] an equation was obtained for the distribution function of a system of charged particles in an electromagnetic field. A relativistic dispersion relation for the transverse and longitudinal waves was found in the self-consistent approximation for the distribution function of eight variables. We found relativistic equations for the quantum distribution functions for scalar charged particles and for electrons. A relativistic quantum equation with a self-consistent field was considered for scalar charged particles. The normalization condition for the distribution function, given in reference 2, may be replaced by a simpler one in those cases when the rest mass of the system of particles under consideration remains constant.

Belyaev and Budker[3] considered, for particles with a Coulomb interaction, a relativistic transport equation which was a generalization of the well-known Landau transport equation. On the basis of that equation they considered the energy and momentum transfer problem from one gas to another.

In Clemmow's and Willson's papers[4] a relativistic transport equation was obtained for the basic distribution function of seven variables $F(t, \mathbf{q}, \mathbf{p})$, without taking collisions into account. A relativistic dispersion equation for longitudinal plasma oscillations was found and studied.

Since equations for different distribution functions are used in the papers mentioned, we first establish a connection between the equations and distribution functions used in the various papers by using as a simple example the transport equation for the distribution function of the variables of one charged particle.

1. RELATIVISTIC DISTRIBUTION FUNCTIONS

When a charged particle moves in an electromagnetic field the square of the four-momentum

$$G = \sum_i \left(P_i - \frac{e}{c} A_i\right)^2 = -m_0^2 c^2.$$

is an integral of motion. Here P_i and A_i are the four-momentum and four-potential. Using this expression for G we obtain the relativistic equation of motion for a charged particle:[2]

$$\frac{dq_i}{ds} = \frac{1}{2m_0}\frac{\partial G}{\partial P_i} = \frac{1}{m_0}\left(P_i - \frac{e}{c}A_i\right) = u_i,$$

$$\frac{dP_i}{ds} = -\frac{1}{2m_0}\frac{\partial G}{\partial q_i} = -\frac{1}{2m_0}\frac{\partial}{\partial q_i}\left(P_i - \frac{e}{c}A_i\right)^2,$$

$$ds = dt\sqrt{1-\beta^2}. \tag{1}$$

We introduce a distribution function of eight variables $f(q_i, P_i)$, defined in such a way that the four vector

$$J_i = \int u_i f(q_i, P_i)\, d^4P$$

is the same as the particle-current four vector.[3] We can then, in the case of a constant rest mass, write down for the distribution function the equation of continuity in (q_i, P_i) space:[2]

$$\frac{\partial}{\partial q_i}\left(\frac{dq_i}{ds}f\right) + \frac{\partial}{\partial P_i}\left(\frac{dP_i}{ds}f\right) = 0. \tag{2}$$

If we integrate (2) over the spatial coordinates and over all momentum components we get

$$\frac{\partial}{\partial q_4}\int \frac{dq_4}{ds} f d^3q d^4p = \frac{\partial}{\partial t}\int \frac{(E-e\varphi)}{m_0c^2} f d^3q d^4P = 0,$$

and we can therefore consider the expression

$$f\frac{u_4}{ic}\, d^3q d^4P = f\frac{(E-e\varphi)}{m_0c^2}\, d^3q d^4P \tag{3}$$

as the probability that the particle finds itself at time t in the spatial volume element d^3q around the point q and has values of its four-momentum in the region d^4P around the point P_i; it is normalized to unity.

We can write (3) in a more symmetric form if we introduce the four-vector of the hypersurface element dS_i. We can then write instead of (3) the more general expression $f\,|(u_i/c)\,dS_i|\,d^4P$ which determines the probability that the world line of the particle intersects the hypersurface element dS_i and that the four-momentum has a value in the region d^4P around P_i. Such a definition was given in a paper by Stratonovich.* Choosing different orientations of the hypersurface element we can obtain instead of (3) other particular definitions.

The equations given above are the same as the ones given by Belyaev and Budker[3] with the only difference that in reference 3 the function $\mathcal{H} = -\sqrt{G}$ was used instead of the function G.

It is possible to define the probability for a state by expression (3) because Eqs. (2) are written for the case where the distribution function does not depend explicitly on a parameter characterizing the trajectory of the motion of the particle in four-dimension space (world line). One can choose the proper time as such a parameter. This problem will be considered in the following in more detail in connection with the quantum generalization of the results stated.

It is in many cases convenient to use instead of a distribution function for the variables q_i and P_i a distribution function for the variables q_i and $p_i = P_i - (e/c)A_i$. The equations of motion take in that case the usual form

$$dq_i/ds = u_i = p_i/m_0, \quad dp_i/ds = (e/c)F_{ik}u_k.$$

We shall write down the corresponding equation of continuity in these variables. Its three-dimensional form is

$$\frac{\mathcal{E}}{m_0c^2}\frac{\partial f}{\partial t} + \mathbf{u}\frac{\partial f}{\partial \mathbf{q}} + e\frac{\partial}{\partial \mathbf{p}}\left\{\gamma\mathbf{E} + \frac{1}{c}[\mathbf{u}\times\mathbf{H}]\right\}f + e(\mathbf{uE})\frac{\partial f}{\partial \mathcal{E}} = 0. \tag{4}$$

Here

$$\mathcal{E} = E - e\varphi, \quad \gamma = \sqrt{1-u^2/c^2}, \quad u = p/m_0.$$

In these variables we get instead of (3) the expression $f(\mathcal{E}/m_0c^2)\,d^3qd^4p$.

Because the mass of the particle is constant, the relation $\Sigma p_i^2 = -m_0^2c^2$ exists between the components of the four-momentum p_i. This makes it possible to simplify (2) and (4) by integrating, for example, over p_4 and reducing them to the equation for the distribution function of seven variables $F(t, \mathbf{q}, \mathbf{p})$. To do this we first clarify the meaning of this distribution function.

Any element of the pseudo-sphere $\Sigma p_i^2 = -m_0^2c^2$ is perpendicular to the vector p_i. If we denote the vector of the hypersurface element in q_i-space by dS_i and in p_i-space by $dS_i^{(p)}$, we have $dS_i^{(p)} \parallel p_i$. The probability that the world line passes through a hypersurface element dS_i while the values of p_i lie within the hypersurface region $dS_i^{(p)}$ will therefore be proportional to the scalar product $dS_i dS_i^{(p)}$. If the vector of the hypersurface element dS_i is in the direction of the time axis, we get for the probability for a state the expression $F(\mathbf{q}, \mathbf{p}, t)\,d^3qd^3p$, where $F(\mathbf{q}, \mathbf{p}, t)$ is an invariant function of seven variables. Such a definition of probability was used by Clemmow and Willson[4] and in earlier papers (see Synge's book[5]). We shall find a connection between the functions $f(q_i, p_i)$ and $F(\mathbf{q}, \mathbf{p}, t)$ and obtain an equation for $F(\mathbf{q}, \mathbf{p}, t)$, starting from Eq. (4).

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*I express my thanks to R. L. Stratonovich for acquainting me with his unpublished work.

Taking it into account that only states lying on the surface $\Sigma p_i^2 = -m_0^2c^2$ are possible, we introduce a function $F(q_i, p_i)$ such that

$$f(q_i, p_i)(\mathscr{E}/m_0c^2)d^3qd^4p$$
$$= F(q_i, p_i)\delta(\mathscr{E} - c\sqrt{m_0^2c^2 + p^2})\, d^3qd^3pd\mathscr{E}.$$

From the expressions given it follows that

$$F(\mathbf{q}, \mathbf{p}, t) = \int F(q_i, p_i)\,\delta(\mathscr{E} - c\sqrt{m_0^2c^2 + p^2})\, d\mathscr{E}. \quad (5)$$

As a result we obtain the following equation

$$\frac{\partial F}{\partial t} + \mathbf{v}\frac{\partial F}{\partial \mathbf{q}} + e\left\{\mathbf{E} + \frac{1}{c}[\mathbf{v}\times\mathbf{H}]\right\}\frac{\partial F}{\partial \mathbf{p}} = 0. \quad (6)$$

This equation is the same as the one given in the papers by Clemmow and Willson.[4] If we go over from the variables $\mathbf{q}, \mathbf{p}, t$ to the variables $\mathbf{q}, \mathbf{v}, t$, we get for the probability of finding the particle in a volume element of phase space the expression[5]*

$$F(t, \mathbf{q}, \mathbf{v})\, m_0^3\gamma^5 d^3qd^3v.$$

Using the distribution function $F(t, \mathbf{q}, \mathbf{p})$ we obtain expressions for the density and the current from the formulae

$$\rho = e\int F(\mathbf{q}, \mathbf{p}, t)\, d^3p, \qquad \mathbf{j} = e\int \mathbf{v}F(\mathbf{q}, \mathbf{p}, t)\, d^3p. \quad (7)$$

We must note that in the general case the equation for one distribution function is insufficient to describe relativistic processes. Indeed, when introducing the function $F(q_i, p_i)$, one must take into account the possibility of states with negative energy values, i.e., put

$$f(q_i, p_i) = F(q_i, p_i)\frac{m_0c^2}{\mathscr{E}}\left\{\delta(\mathscr{E} - c\sqrt{m_0^2c^2 + p^2}) + \delta(\mathscr{E} + c\sqrt{m_0^2c^2 + p^2})\right\}.$$

We get in that case instead of one Eq. (6) a set of equations for two functions F^+ and F^-:

$$\pm\frac{\partial F^\pm}{\partial t} + \mathbf{v}\frac{\partial F^\pm}{\partial \mathbf{q}} + e\left\{\mathbf{E} + \frac{1}{c}[\mathbf{v}\times\mathbf{H}]\right\}\frac{\partial F^\pm}{\partial \mathbf{p}} = 0. \quad (8)$$

The positive sign corresponds to a positive energy value and the minus sign to a negative one. The equations for F^+ and F^- differ only by the sign in front of the first term. We shall consider (8) in more detail in the following in connection with the derivation of transport equations that take pair production into account.

*A relativistic transport equation with a self-consistent field for the function $F(\mathbf{q}, \mathbf{v}, t)$ was also considered in A. A. Vlasov's book Теория многих частиц (Many-Particle Theory), Gostekhizdat, 1950.

2. A CHAIN OF EQUATIONS FOR THE RELATIVISTIC DISTRIBUTION FUNCTIONS

There are two possible ways of obtaining chains of equations for the relativistic distribution functions. One is a direct generalization of Bogolyubov's well-known method to the relativistic case. To do this we must consider distribution functions both of the particle variables and of the coordinates and the momenta of the field oscillators. Some non-relativistic problems with such distribution functions were considered in references 6 – 8.

We shall obtain the chain of equations by a slightly different method, based upon the use of a set of equations for the random function $N_{\mathbf{qp}}(t)$ and the electrical and magnetic field strengths, which are also considered as random functions. The function $N_{\mathbf{qp}}(t)$ is defined in such a way that the expression

$$N_{\mathbf{qp}}d^3qd^3p = \sum_{1\leqslant l\leqslant N} \delta(\mathbf{q} - \mathbf{q}^{(l)})\,\delta(\mathbf{p} - \mathbf{p}^{(l)}) \quad (9)$$

determines the number of particles at time t in a volume element of phase space d^3qd^3p around the point $\mathbf{q}, \mathbf{p}$. Equations for the random functions $N_{\mathbf{qp}}$ for systems of particles with a central force interaction were considered, for instance, in references 9 – 11. A set of equations for the function $N_{\mathbf{qp}}$ and the vector and scalar potential served in a paper by the author[12] as the starting point for considering the space-time correlation functions for a system of charged particles with an electromagnetic interaction. We shall use the results of that paper to a large extent in this section.

If we can forget about states with a negative energy, the equation for the function $N_{\mathbf{qp}}(t)$ is formally the same as (6):

$$\frac{\partial N_{\mathbf{qp}}}{\partial t} + \mathbf{v}\frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{q}} + e\left\{\mathbf{E} + \frac{1}{c}[\mathbf{v}\times\mathbf{H}]\right\}\frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{p}} = 0. \quad (9')$$

In order that the set of equations be complete, we must still add the equations for the electromagnetic field

$$\operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + 4\pi e\int \mathbf{v}N_{\mathbf{qp}}d^3p, \qquad \operatorname{div}\mathbf{H} = 0,$$
$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \qquad \operatorname{div}\mathbf{E} = 4\pi e\left\{\int N_{\mathbf{qp}}d^3p - n_0\right\}. \quad (10)$$

The set (9) and (10) is also a closed set of relativistic equations for the random functions $N_{\mathbf{qp}}$, $\mathbf{E}$, and $\mathbf{H}$, which describe different states of the plasma electrons and the electric and magnetic field strengths. In the following we assume that the role of the positively charged ions is reduced to the role of a background which compensates the charge of the electrons. One can consider the more general case in a similar manner.

If we take states with a negative energy into account we must introduce two random functions $N^{\pm}_{\mathbf{qp}}(t)$, the equations for which are formally the same as (8). In a corresponding fashion the expressions for the density and the current on the right-hand sides of (10) are also changed.

As an initial set of equations we can use the equation for the random function of eight variables q_i, p_i, defining it in such a way that the expression

$$N_{q_i, p_i}\gamma d^3q d^4p = \sum_{1\leqslant l\leqslant N} \frac{1}{\gamma^{(l)}} \delta(\mathbf{q} - \mathbf{q}^{(l)}) \delta(p_i - p_i^{(l)}) \gamma d^3q d^4p \tag{11}$$

determines the number of particles whose world lines intersect an element of hypersurface oriented along the time axis while their momenta lie in the region d^4p around the point p_i.

We can write expression (11) in the explicit relativistically-invariant form proposed by Stratonovich:

$$N_{q_i p_i} = \sum_{1\leqslant l\leqslant N} \int \delta(q_i - q_i^{(l)}(s^{(l)})) \delta(p_i - p_i^{(l)}(s^{(l)})) ds^{(l)}.$$

Taking into account that $ds = dt/\gamma$ and performing an integration in this expression, we get Eq. (11).

The equation of motion for the functions $N_{q_i P_i}$ has the following form in the variables q_i, P_i

$$(\partial G / \partial P_i) \partial N_{q_i P_i} / \partial q_i - (\partial G / \partial q_i) \partial N_{q_i P_i} / \partial P_i = 0,$$

and in the variables q_i, p_i

$$u_i \frac{\partial}{\partial q_i} N_{q_i p_i} + \frac{e}{c} F_{ik} u_k \frac{\partial}{\partial p_i} N_{q_i p_i} = 0. \tag{12}$$

Together with the equations

$$\partial F_{ik} / \partial q_k = 4\pi e \int u_i N_{q_i p_i} d^4p,$$

$$\partial F_{ik} / \partial q_l + \partial F_{kl} / \partial q_i + \partial F_{li} / \partial q_k = 0 \tag{13}$$

(12) forms a closed set of equations for the random function $N_{q_i p_i}$ and the tensor F_{ik} whose components are also considered to be random functions.

Using the set (9) and (10) [or (12) and (13)] we can obtain a chain of equations for the moments of the random functions or for the corresponding distribution functions. This problem has been considered before[12] for a non-relativistic plasma. There we obtained expressions for the space-time correlation functions of the charge and current density, and for the values of the vector and scalar potentials. Using (9) and (10) we can obtain similar results for a relativistic plasma.

We shall consider a more general approach, which enables us to take directly into account the thermal motion of the electromagnetic field. We determine the state of the system by giving the coordinates and momenta of the particles $\mathbf{q}_1, \ldots, \mathbf{q}_N$, $\mathbf{p}_1, \ldots, \mathbf{p}_N$ and the coordinates and momenta of the oscillators of the transverse electromagnetic field $Q_{\mathbf{k}}^{(j)}$ and $P_{\mathbf{k}}^{(j)}$ $(j = 1, 2)$.

The equations of motion for these quantities can be written in the following form in the Coulomb gauge[13]

$$\dot{\mathbf{q}}^{(l)} = \mathbf{v}^{(l)}, \quad \dot{\mathbf{p}}^{(l)} = -\frac{\partial}{\partial \mathbf{q}^{(l)}} \sum_{1\leqslant l\leqslant N} \Phi(|\mathbf{q}^{(l)} - \mathbf{q}^{(l')}|) - \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{e}{c}[\mathbf{v} \times \operatorname{curl} \mathbf{A}]; \tag{14}$$

$$\dot{Q}_{\mathbf{k}}^{(j)} = P_{\mathbf{k}}^{(j)},$$

$$\dot{P}_{\mathbf{k}}^{(j)} = -\omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^{(j)} + e\sqrt{4\pi/V} \sum_{1\leqslant l\leqslant N} (\mathbf{v}\mathbf{a}_{\mathbf{k}}) {}^{\sin}_{\cos} \mathbf{k}\mathbf{q}^{(l)}. \tag{15}$$

In those equations $\mathbf{a}_{\mathbf{k}}$ is a unit vector in the direction of the component of a vector-potential with wave vector $\mathbf{k}$. Then

$$\mathbf{A} = \sqrt{4\pi c^2/V} \sum_{\mathbf{k}} \mathbf{a}_{\mathbf{k}} (Q_{\mathbf{k}}^{(1)} \sin \mathbf{kq} + Q_{\mathbf{k}}^{(2)} \cos \mathbf{kq}), \quad \Phi(q) = e^2/q.$$

The connection between the velocity and the momentum of a particle is of the form $\mathbf{v} = \gamma \mathbf{p}/m_0$.

Together with the random functions $N_{\mathbf{qp}}(t)$ we introduce another random function defined in such a way that the expression

$$N_{Q_{\mathbf{k}}^{(j)} P_{\mathbf{k}}^{(j)}} dQ_{\mathbf{k}}^{(j)} dP_{\mathbf{k}}^{(j)} \text{ (or more briefly, } N_{Q_{\mathbf{k}} P_{\mathbf{k}}} dQ_{\mathbf{k}}^{(j)} dP_{\mathbf{k}}^{(j)})$$

is the number of oscillators of the transverse electromagnetic field with wave vector $\mathbf{k}$ whose coordinates and momenta lie at the time t in the region $dQ_{\mathbf{k}}^{(j)} dP_{\mathbf{k}}^{(j)}$ around the point $Q_{\mathbf{k}}^{(j)} P_{\mathbf{k}}^{(j)}$.

We can write down the equations for the random functions introduced in this way, using the equations of motion (14) and (15)

$$\frac{\partial N_{\mathbf{qp}}}{\partial t} + \mathbf{v}\frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}} \int \Phi(|\mathbf{q} - \mathbf{q}'|) N_{\mathbf{q}'\mathbf{p}'} d\mathbf{q}' d\mathbf{p}' \frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{p}} + \frac{e}{c} \int \left(-\frac{\partial \mathbf{A}}{\partial t} + [\mathbf{v} \times \operatorname{curl} \mathbf{A}]\right) N_{Q_{\mathbf{k}} P_{\mathbf{k}}} dQ_{\mathbf{k}}^{(j)} dP_{\mathbf{k}}^{(j)} \frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{p}} = 0; \tag{16}$$

$$\frac{\partial}{\partial t} N_{Q_{\mathbf{k}} P_{\mathbf{k}}} + \sum_j \left(P_{\mathbf{k}}^{(j)} \frac{\partial}{\partial Q_{\mathbf{k}}^{(j)}} N_{Q_{\mathbf{k}} P_{\mathbf{k}}} - \omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^{(j)} \frac{\partial}{\partial P_{\mathbf{k}}^{(j)}} N_{Q_{\mathbf{k}} P_{\mathbf{k}}}\right) + e\sqrt{\frac{4\pi}{V}} \int (\mathbf{v}'\mathbf{a}_{\mathbf{k}}) {}^{\sin}_{\cos} \mathbf{k}\mathbf{q}' N_{\mathbf{q}'\mathbf{p}'} d\mathbf{q}' d\mathbf{p}' \frac{\partial}{\partial P_{\mathbf{k}}^{(j)}} N_{Q_{\mathbf{k}} P_{\mathbf{k}}} = 0. \tag{17}$$

We must take into account in the first equation that

$$A = \sqrt{4\pi c^2/V} \sum_{k,j} Q_{\mathbf{k}}^{(j)} {}^{\sin}_{\cos} \mathbf{kq}.$$

One can use the set of relativistic equations (16) and (17) to obtain a chain of equations both for single-time and for many-time distribution functions.

We shall take into consideration the connection between the distribution functions and the moments of the random functions

$$\overline{N_{\mathbf{qp}}(t)\,N_{\mathbf{q}'\mathbf{p}'}(t)} = V^{-2}N(N-1)\,f_2(\mathbf{q},\mathbf{p},\mathbf{q}',\mathbf{p}',t) + (N/V)\,\delta(\mathbf{q}-\mathbf{q}')\,\delta(\mathbf{p}-\mathbf{p}')\,f_1(\mathbf{q},\mathbf{p},t),$$

$$\overline{N_{\mathbf{qp}}(t)\,N_{\mathbf{q}'\mathbf{p}'}(t)\,N_{\mathbf{q}''\mathbf{p}''}(t)} = V^{-3}N(N-1)(N-2)\,f_3(\mathbf{q},\mathbf{p},\mathbf{q}',\mathbf{p}',\mathbf{q}'',\mathbf{p}'',t)$$
$$+ V^{-2}N(N-1)\{\delta(\mathbf{q}-\mathbf{q}')\,\delta(\mathbf{p}-\mathbf{p}')\,f_2(\mathbf{q},\mathbf{p},\mathbf{q}'',\mathbf{p}'',t)$$
$$+ \delta(\mathbf{q}-\mathbf{q}'')\,\delta(\mathbf{p}-\mathbf{p}'')\,f_2(\mathbf{q},\mathbf{p},\mathbf{q}',\mathbf{p}',t)$$
$$+ \delta(\mathbf{q}'-\mathbf{q}'')\,\delta(\mathbf{p}'-\mathbf{p}'')\,f_2(\mathbf{q},\mathbf{p},\mathbf{q}',\mathbf{p}',t)\}$$
$$+ (N/V)\,\delta(\mathbf{q}-\mathbf{q}')\,\delta(\mathbf{p}-\mathbf{p}')$$
$$\times\,\delta(\mathbf{q}-\mathbf{q}'')\,\delta(\mathbf{p}-\mathbf{p}'')\,f_1(\mathbf{q},\mathbf{p},t),$$

$$\overline{N_{\mathbf{qp}}(t)\,N_{\mathbf{q}'\mathbf{p}'}(t)\,N_{Q_\mathbf{k}P_\mathbf{k}}(t)} = V^{-2}N(N-1)\,\Phi_3(\mathbf{q},\mathbf{p},\mathbf{q}',\mathbf{p}',Q_\mathbf{k},P_\mathbf{k},t)$$
$$+ (N/V)\,\delta(\mathbf{q}-\mathbf{q}')\,\delta(\mathbf{p}-\mathbf{p}')\,\Phi_2(\mathbf{q},\mathbf{p},Q_\mathbf{k},P_\mathbf{k},t).$$

In these equations f_1, f_2, and f_3 are the first, second, and third distribution functions of the electrons, normalized in such a way that $V^{-s}\int f_s d\mathbf{q}d\mathbf{p} = 1$, and Φ_2 and Φ_3 are the second and third compound distribution functions for the electrons and the oscillators of the electromagnetic field. Using the expressions that follow from (16) and (17) we get, after averaging, the following equations for the first distribution functions for the electrons and for the oscillators $f_1(\mathbf{q},\mathbf{p},t)$, $F_1(Q_\mathbf{k}^{(j)}, P_\mathbf{k}^{(j)}, t)$

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\frac{\partial f_1}{\partial \mathbf{q}} - n\frac{\partial}{\partial \mathbf{q}}\int \Phi(|\mathbf{q}-\mathbf{q}'|)\frac{\partial}{\partial \mathbf{p}} f_2 dq'dp'$$
$$+ \frac{e}{c}\int\left(-\frac{\partial \mathbf{A}}{\partial t} + [\mathbf{v}\times\mathbf{curl}\,\mathbf{A}]\right)$$
$$\times\frac{\partial}{\partial \mathbf{p}}\Phi_2(\mathbf{q},\mathbf{p},Q_\mathbf{k}^{(j)},P_\mathbf{k}^{(j)},t)\,dQ_\mathbf{k}^{(j)}dP_\mathbf{k}^{(j)} = 0, \tag{18}$$

$$\frac{\partial F_1}{\partial t} + \sum_j\left(P_\mathbf{k}^{(j)}\frac{\partial}{\partial Q_\mathbf{k}^{(j)}} - \omega_\mathbf{k}^2 Q_\mathbf{k}^{(j)}\frac{\partial}{\partial P_\mathbf{k}^{(j)}}\right)$$
$$\times F_1 + en\sqrt{\frac{4\pi}{V}}\int(\mathbf{v}'\mathbf{a}_\mathbf{k})\,{}^{\sin}_{\cos}\,\mathbf{kq}'\,\frac{\partial}{\partial P_\mathbf{k}^{(j)}}\Phi_2 dq'dp' = 0. \tag{19}$$

The single-time second distribution functions f_2 and Φ_2 for which we can obtain equations by similar means, enter into these equations, only the single-time third distribution functions enter in the equations for the second distribution functions, and so on.

The initial set (16) and (17) can be written in relativistically invariant form if we use instead of the equation for $N_{\mathbf{qp}}(t)$ the equation for the function $N_{q_ip_i}$ and the corresponding relativistically invariant equation for the function that determines the number of field oscillators.

The chain of equations deduced here can be obtained from the equation for the distribution function for the coordinates and momenta of all particles and field oscillators at one and the same time

$$f(\mathbf{q}_1,\ldots,\mathbf{q}_N,\,\mathbf{p}_1,\ldots,\mathbf{p}_N,\ldots,Q_\mathbf{k}^{(j)},\ldots,P_\mathbf{k}^{(j)},\ldots,t).$$

The equation for this distribution function is of the following form

$$\frac{\partial f}{\partial t} + \sum_{1\le l\le N}\mathbf{v}^{(l)}\frac{\partial f}{\partial \mathbf{q}^{(l)}} + \sum_{1\le l\le N}\left\{-\frac{\partial f}{\partial \mathbf{q}^{(l)}}\sum_{l'}\Phi(|\mathbf{q}^{(l)}-\mathbf{q}^{(l')}|)\right.$$
$$\left.+\frac{e}{c}\left(-\frac{\partial \mathbf{A}}{\partial t} + [\mathbf{v}^{(l)}\times\mathbf{curl}\,\mathbf{A}]\right)\right\}\frac{\partial f}{\partial \mathbf{p}^{(l)}}$$
$$+\sum_{k,j}\left(P_\mathbf{k}^{(j)}\frac{\partial f}{\partial Q_\mathbf{k}^{(j)}} - \omega_\mathbf{k}^2 Q_\mathbf{k}^{(j)}\frac{\partial f}{\partial P_\mathbf{k}^{(j)}}\right)$$
$$+ e\sqrt{\frac{4\pi}{V}}\sum_{k,j,l}(\mathbf{v}^{(l)}\mathbf{a}_\mathbf{k})\,{}^{\sin}_{\cos}\,\mathbf{kq}^{(l)}\frac{\partial f}{\partial P_\mathbf{k}^{(j)}} = 0. \tag{20}$$

3. A VARIATIONAL PRINCIPLE FOR A RELATIVISTIC PLASMA

It is well known that the action function for a system of charged particles and the electromagnetic field can be written, for instance, in the form[14]

$$S = \sum_{1\le l\le N}\left\{-m_0c\int\sqrt{-u_i^{(l)2}}\,ds^{(l)} + \frac{e}{c}\int A_iu_i^{(l)}ds^{(l)}\right\}$$
$$+\frac{i}{16\pi c}\int F_{ik}^2 d^4q.$$

Using the function $N_{q_ip_i}$ and taking into account that $ds = dt/\gamma$ and $dq_4 = icdt$ we can write this expression in the form

$$S = \frac{1}{ic}\int\left\{-m_0c\sqrt{-u_i^2} + \frac{e}{c}A_iu_i\right\}N_{q_ip_i}d^4qd^4p$$
$$+\frac{i}{16\pi c}\int F_{ik}^2 d^4q. \tag{21}$$

Using the latter expression we determine the equations of motion in the following way

$$P_i = \frac{\partial}{\partial u_i}\frac{\delta S}{\delta N_{q_ip_i}} = p_i + \frac{e}{c}A_i,$$

$$\dot{P}_i = \frac{\partial}{\partial q_i}\frac{\delta S}{\delta N_{q_ip_i}} = \frac{e}{c}\frac{\partial}{\partial q_i}(A_lu_l).$$

Taking into account that

$$dA_i/ds = u_k\partial A_i/\partial q_k,$$

$$\partial(A_ku_k)/\partial q_i = u_k\partial A_i/\partial q_k + F_{ik}u_k,$$

we get the equations of motion in the usual form

$$m_0 du_i / ds = (e/c) F_{ik} u_k.$$

The equations for the field are obtained as usual by varying the vector potential and are the same as (13), while the equation for the function $N_{q_i p_i}$ itself, which follows from the equations of motion is the same as (12).

We shall average expression (21) for the action function over the whole system; such an average will be denoted by a superior bar. This average value of the action function is determined not only by the average values $\overline{N_{q_i p_i}}$, $\overline{A_i}$, $\overline{F_{ik}}$, but also by the values of the correlation functions of the field and particle variables and of the field fluctuations. We shall introduce the deviations from the average values $\delta N_{q_i p_i}$ and δF_{ik}. We have then

$$\overline{A_i N_{q_i p_i}} = \overline{A}_i \overline{N}_{q_i p_i} + \overline{\delta A_i \delta N_{q_i p_i}}, \qquad \overline{F^2_{ik}} = \overline{F}^2_{ik} + \overline{\delta F^2_{ik}}.$$

If we neglect the correlation functions $\overline{\delta A_i \delta N_{q_i p_i}}$ and the field fluctuations, the expression for $\overline{S}$ takes the form

$$\overline{S} = \frac{1}{ic}\int\left(-m_0 c\sqrt{-u_i^2} + \frac{e}{c}\overline{A}_i u_i\right)\overline{N}_{q_i p_i} d^4q d^4p + \frac{i}{16\pi c}\int \overline{F}^2_{ik}\, d^4q.$$

This relation can be used to derive the relativistic equations for the particle distribution function $\overline{N_{q_i p_i}}$ and the components of the tensor $\overline{F_{ik}}$ with a self-consistent field (without taking correlation into account).

If we introduce instead of the function $N_{q_i p_i}$ the random function $N_{\mathbf{qp}}(t)$, the average value of which is the normal distribution function of seven variables **q**, **p**, and t, we are led to the following expression for S after integrating over the energy

$$S = \int\left\{-m_0 c^2 + \frac{e}{c} A_i u_i\right\} N_{qp} d^3q d^3p ds + \frac{i}{16\pi c}\int F^2_{ik} d^4q.$$

In the non-relativistic approximation this expression takes the form

$$S = \int\left\{\frac{mv^2}{2} + \frac{e}{c}\mathbf{A}\mathbf{v} - e\varphi\right\} N_{\mathbf{qp}} d^3q d^3p dt + \frac{1}{8\pi}\int (\mathbf{E}^2 - \mathbf{H}^2)\, d^3q dt, \qquad \mathbf{v} = \frac{\mathbf{p}}{m}.$$

If we average the last expression, neglecting correlation and field fluctuations, we get an approximate expression for the average value of the action function

$$\overline{S} = \int\left\{\frac{mv^2}{2} + \frac{e}{c}\overline{\mathbf{A}}\mathbf{v} - e\overline{\varphi}\right\}\overline{N}_{\mathbf{qp}} d^3q d^3p dt + \frac{1}{8\pi}\int (\mathbf{E}^2 - \mathbf{H}^2)\, d^3q dt.$$

This equation is the same as the expression for the action function used in Low's work.[15]

4. RELATIVISTIC EQUATIONS WITH A SELF-CONSISTENT FIELD

In the present section we consider the relativistic transport equation in the self-consistent field approximation.

We first consider the set (9) and (10). We introduce functions that characterize the deviations of the random functions from their average values $\delta N_{\mathbf{qp}}$, $\delta\mathbf{E}$, and $\delta\mathbf{H}$, and express the second moments of the random functions in terms of the central moments; for instance,

$$\overline{EN_{\mathbf{qp}}} = \overline{E}\,\overline{N}_{\mathbf{qp}} + \overline{\delta E \delta N_{\mathbf{qp}}}.$$

If we break off the chain of equations right at the beginning by neglecting the second central moments we obtain a set of relativistic self-consistent equations

$$\frac{\partial \overline{N}_{\mathbf{qp}}}{\partial t} + \mathbf{v}\frac{\partial \overline{N}_{\mathbf{qp}}}{\partial \mathbf{q}} + e\left(\overline{\mathbf{E}} + \frac{1}{c}[\mathbf{v}\times\overline{\mathbf{H}}]\right)\frac{\partial \overline{N}_{\mathbf{qp}}}{\partial \mathbf{p}} = 0, \qquad p = m_0 v\gamma;$$

$$\operatorname{curl}\overline{\mathbf{H}} = \frac{1}{c}\frac{\partial \overline{\mathbf{E}}}{dt} + 4\pi e\int \mathbf{v}\overline{N}_{\mathbf{qp}} d^3p, \qquad \operatorname{div}\overline{\mathbf{H}} = 0;$$

$$\operatorname{curl}\overline{\mathbf{E}} = -\frac{1}{c}\frac{\partial \overline{\mathbf{H}}}{\partial t}, \qquad \operatorname{div}\overline{\mathbf{E}} = 4\pi e\left\{\int \overline{N}_{\mathbf{qp}} d^3p - n_0\right\}. \quad (22)$$

This set of equations differs only by the relativistic dependence of the momentum on the velocity from the classical set of self-consistent field equations for a plasma, first considered by Vlasov.[16]

We give here expressions for the dispersion relations for transverse and longitudinal plasma waves obtained under the assumption that the distribution function $\overline{N}_{\mathbf{qp}}(t) = nf_1(\mathbf{q}, \mathbf{p}, t)$ differs little from the stationary uniform distribution for which we must take in this case the relativistic Maxwell distribution

$$\overline{N}_{\mathbf{p}} = A\exp\{-c\sqrt{m_0^2c^2 + p^2}/\varkappa T\}.$$

The relativistic dispersion relations have the following form

$$1 = \frac{4\pi e^2\omega}{\varkappa T(\omega^2 - c^2k^2)}\int\frac{(\gamma u_y)^2\overline{N}_{\mathbf{p}}}{\omega - \gamma u_x k} d^3p$$

for transverse oscillations and

$$1 = \frac{4\pi e^2}{\varkappa T\omega}\int\frac{(\gamma u_x)^2\overline{N}_{\mathbf{p}}}{\omega - \gamma u_x k} d^3p$$

for longitudinal oscillations. The last expression is the same as the dispersion relation given by Clemmow and Willson.[4]

The set of self-consistent equations for the distribution function $\overline{N}_{q_i p_i}$ has the following form

$$u_i \partial \overline{N}_{q_i p_i} / \partial q_i + \frac{e}{c} F_{ik} u_k \frac{\partial}{\partial p_i} \overline{N}_{q_i p_i} = 0,$$

$$\partial \overline{F}_{ik} / \partial q_k = 4\pi e \int u_i \overline{N}_{q_i p_i} d^4 p,$$

$$\partial F_{ik} / \partial q_l + \partial F_{kl} / \partial q_i + \partial F_{li} / \partial q_k = 0.$$

The relativistically invariant dispersion relations had been obtained earlier.[2]

By averaging (16) and (17) or by using the first equations of the chain for the distribution functions [Eqs. (18) and (19)] we can obtain a more general set of self-consistent equations for the distribution functions of the electrons and the field oscillators, taking the thermal motion of the electromagnetic field into account. This set of equations has the following form:

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \frac{\partial f_1}{\partial \mathbf{q}} - n \frac{\partial}{\partial \mathbf{q}} \int \Phi (|\mathbf{q} - \mathbf{q}'|) f_1 d\mathbf{q}' d\mathbf{p}' \frac{\partial f_1}{\partial \mathbf{p}}$$

$$+ \frac{e}{c} \int \left(-\frac{\partial \mathbf{A}}{\partial t} + [\mathbf{v} \times \mathrm{curl}\, \mathbf{A}] \right)$$

$$\times F_1 (Q_{\mathbf{k}}^{(j)} P_{\mathbf{k}}^{(j)} t)\, dQ_{\mathbf{k}}^{(j)} dP_{\mathbf{k}}^{(j)} \frac{\partial f_1}{\partial \mathbf{p}} = 0,$$

$$\frac{\partial F_1}{\partial t} + \sum_j \left(P_{\mathbf{k}}^{(j)} \frac{\partial F_1}{\partial Q_{\mathbf{k}}^{(j)}} - \omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^{(j)} \frac{\partial F_1}{\partial P_{\mathbf{k}}^{(j)}} \right)$$

$$+ en \sqrt{\frac{4\pi}{V}} \int (\mathbf{v}' \mathbf{a}_{\mathbf{k}}) {\sin \atop \cos} \mathbf{k}\mathbf{q}' f_1 d\mathbf{q}' d\mathbf{p}' \frac{\partial F_1}{\partial P_{\mathbf{k}}^{(j)}} = 0.$$

In forthcoming parts of the present paper we shall consider more accurate transport equations and equations for the correlation functions.

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[1] N. N. Bogolyubov, Проблемы динамической теории в статистической физике, (Problems of Dynamic Theory in Statistical Physics), Gostekhizdat, 1946.

[2] Yu. L. Klimontovich, Dokl. Akad. Nauk SSSR **87**, 927 (1952); Тр. Московского авиационного технологич. ин-та (Proc. Moscow Aviation Technological Inst.) No. 26 (1955).

[3] S. T. Belyaev and G. I. Budker, Dokl. Akad. Nauk SSSR **107**, 807 (1956), Soviet Phys.–Doklady **1**, 218 (1956).

[4] P. C. Clemmow and A. J. Willson, Proc. Roy. Soc. (London) **A237**, 117 (1956); Proc. Cambridge Phil. Soc. **53**, 222 (1957).

[5] J. L. Synge, The Relativistic Gas (North Holland Publ. Co., Amsterdam, 1957).

[6] W. E. Brittin, Phys. Rev. **106**, 843 (1957).

[7] R. N. Gurzhi, JETP **33**, 451 (1957), Soviet Phys. JETP **6**, 352 (1958).

[8] Yu. L. Klimontovich, JETP **35**, 1276 (1958), Soviet Phys. JETP **8**, 891 (1959); Yu. L. Klimontovich and S. V. Temko, JETP **35**, 1141 (1958), Soviet Phys. JETP **8**, 799 (1959).

[9] Yu. L. Klimontovich, Dokl. Akad. Nauk SSSR **96**, 43 (1954); JETP **33**, 982 (1957), Soviet Phys. JETP **6**, 753 (1958).

[10] R. L. Stratonovich, Thesis, Moscow State University (1956).

[11] B. B. Kadomtsev, JETP **32**, 943 (1957), Soviet Phys. JETP **5**, 771 (1957); JETP **33**, 151 (1957), Soviet Phys. **6**, 117 (1958).

[12] Yu. L. Klimontovich, JETP **34**, 173 (1958), Soviet Phys. JETP **7**, 119 (1958).

[13] W. Heitler, Quantum Theory of Radiation (Oxford University Press, 1954).

[14] L. D. Landau and E. M. Lifshitz, Теория поля (Field Theory), Gostekhizdat, 1948 [The Classical Theory of Fields, Addison-Wesley, 1951].

[15] F. E. Low, Proc. Roy. Soc. (London) **A248**, 282 (1958).

[16] A. A. Vlasov, Уч. зап. МГУ, (Sci. Notes, Moscow State Univ.) No. 76, part 1, 1944; JETP **8**, 291 (1938).

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