ON THE RELATIVISTIC OPERATORS FOR MOMENTUM AND ANGULAR MOMENTUM

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Correct expressions for the relativistic operators for momentum and angular-momentum components in orthogonal curvilinear coordinates are derived by a systematic application of the theory of spinors.

INTRODUCTION

It is well known that the operators for momentum and for covariant differentiation differ only by a constant factor. Since the Dirac equation contains a linear combination of products of momentum components by a spinor, one must then use covariant derivatives of a spinor to write this equation in curvilinear coordinates. Operators for covariant differentiation of a spinor have been calculated by Fock and by Ivanenko and are given in the books of these authors.[1,2] The virtue of these operators is that they agree with the rule for differentiation of a vector and give the correct result when a certain linear combination of them is substituted in the Dirac equation. Individually, however, these operators are incorrect, as is shown below by a particular example. The purpose of the present paper is to use an elementary but systematic application of the theory of spinors to get the correct expression for the operator of covariant differentiation of a spinor, and along with it the expressions for the momentum and angular-momentum operators.

1. THE COVARIANT DERIVATIVE OF A SPINOR

The spinor ξ is to be regarded as a one-column matrix with four complex components.

It is convenient to represent vectors and bivectors by square matrices, as Cartan[3] does. Thus one puts in correspondence with the vector V^i the matrix $V \equiv V^i H_i$ (summation from 1 to 4), where

$$H_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Any two of the matrices H_i anticommute, and $H_i^2 = e_i$, $e_1 = e_2 = e_3 = 1$, $e_4 = -1$. We shall confine our treatment to orthogonal bases consisting of unit vectors (a Galilean frame) relative to which the components of vectors and bivectors are real, the space axes having the numbers 1, 2, 3, and the time axis being taken as the fourth.

In finding the covariant derivative of a spinor, $D_k\xi$, we follow Cartan in using the following definition of the covariant derivative $D_i f$ of any quantity f:

$$Df = D_i f \omega^i, \tag{2}$$

where Df is the actual increase (the absolute differential) of the quantity f, and ω^i is the i-th Galilean component of the vector connecting two neighboring points:

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 - (\omega^4)^2. \tag{3}$$

The form of the operator D_i depends on the object f to which it is applied. If we denote by ω^{ik} the components of the bivector of an infinitesimal rotation ($\omega^{ii} = 0$) and by Ω the Cartan matrix of this bivector

$$\Omega \equiv \tfrac{1}{2} \omega^{ik} H_i H_k, \tag{4}$$

we have for the absolute differential of a spinor[3]

$$D\xi = d\xi - \tfrac{1}{2}\Omega\xi. \tag{5}$$

The components of the bivector Ω are found by an elementary calculation (cf., e.g., reference 4, p. 217). Let the standard form of the line element in our orthogonal coordinates x^k be

$$ds^2 = \sum_{k=1}^{4} g_{kk} (dx^k)^2, \tag{6}$$

where $x^4 = ct$ (c is the speed of light, t the time). Let us introduce the briefer notations $e_k g_{kk} = g_k$; then

$$\omega^{ik} = \frac{1}{2\sqrt{g_i g_k}} \left(e_i \frac{\partial g_k}{\partial x^i} dx^k - e_k \frac{\partial g_i}{\partial x^k} dx^i \right). \tag{7}$$

From Eqs. (2), (5), and (4) it follows that

$$\sum_{i=1}^{4} D_i \xi \omega^i = d\xi - \frac{1}{4} \sum_{i,k=1}^{4} \omega^{ik} H_i H_k \xi.$$

Inserting the quantities (7), we find

$$\sum_{i=1}^{4} D_i \xi \omega^i = d\xi - \frac{1}{8} \sum_{i,k=1}^{4} \frac{e_i}{\sqrt{g_i g_k}} \frac{\partial g_k}{\partial x^i} H_i H_k \xi dx^k$$

$$+ \frac{1}{8} \sum_{i,k=1}^{4} \frac{e_k}{\sqrt{g_i g_k}} \frac{\partial g_i}{\partial x^k} H_i H_k \xi dx^i.$$

In the first double sum we interchange the summation indices i and k; the right member of the last equation is then written in the form

$$d\xi + \frac{1}{8} \sum_{i,k=1}^{4} \frac{e_k}{\sqrt{g_i g_k}} \frac{\partial g_i}{\partial x^k} (H_i H_k - H_k H_i) \xi dx^i.$$

The terms with $i = k$ cancel. Using the fact that the matrices H_k anticommute, we get finally

$$\sum_{i=1}^{4} D_i \xi \omega^i = d\xi + \frac{1}{4} \sum_{i=1}^{4} \sum_{k \neq i} \frac{e_k}{\sqrt{g_i g_k}} \frac{\partial g_i}{\partial x^k} H_i H_k \xi dx^i, \quad (8)$$

and from a comparison of Eqs. (3) and (6) it follows that

$$\omega^i = \sqrt{g_i}\, dx^i. \quad (9)$$

In virtue of the arbitrariness of the differentials dx^i we get from Eqs. (8) and (9)

$$D_i \xi = \frac{1}{\sqrt{g_i}} \frac{\partial \xi}{\partial x^i} + \frac{1}{4 g_i} \sum_{k \neq i} \frac{e_k}{\sqrt{g_k}} \frac{\partial g_i}{\partial x^k} H_i H_k \xi. \quad (10)$$

This is the expression for the covariant derivative of a spinor prescribed in terms of its components relative to the Galilean coordinate frame x^k. Because of the presence of the matrices the operator D_i is nondiagonal, and this nondiagonality cannot be removed by any kind of unitary transformation.*

2. RELATION OF THE COVARIANT DERIVATIVE TO THE DIRAC EQUATION

Since the basis for the derivation of Eq. (10) is Eq. (2), in which the symbol D has scalar character, it is clear that the index i in the expression $D_i \xi$ is a full-fledged covariant index, and the quantities $D_i \xi$ form a tensor of rank three halves (a spinor being regarded as a tensor of rank one half). Thus we have a right to put in correspondence with the canonical momentum P_k the expression $P_k \to -i\hbar D_k$. The treatments by Fock[1] and by Ivanenko and Sokolov[2] give expressions for the momentum operator in which the index k is not a covariant index. The correctness of the results so obtained is due to a peculiarity of the Dirac equation, which we shall now analyze very briefly.

It is well known that the momentum components P^k, k = 1, 2, 3 and $P^4 = E/c$ (where E is the total energy of the particle) form a contravariant four-vector. The Cartan matrix of this vector is $P = P^k H_k$.

The eigenvalue equation of the energy-momentum operator, written in the form

$$P\xi = \lambda H_0 \xi, \quad (11)$$

gives the value of the spinor for a free electron, at the origin of the coordinates. The matrix

$$H_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is introduced so that the left and right members of Eq. (11) will transform in the same way.[3] For agreement with the theory of relativity we must have $\lambda = \pm m_0 c$, where m_0 is the rest mass of the electron. To get the value of the spinor at the other points of the space, we must make the replacement $P_k \to -i\hbar D_k$, or $P^k \to -i\hbar e_k D_k$, and add the matrix V of the electromagnetic potential, multiplied by e/c (−e is the charge of the electron), if there is an external field. Furthermore one sets $\lambda = +m_0 c$, and the Dirac equation takes the form

$$(-i\hbar \nabla + eV/c - m_0 c H_0)\xi = 0.$$

where

$$\nabla \xi = \sum_{i=1}^{4} e_i H_i D_i \xi$$

Using Eq. (10) and recalling that $H_i^2 = e_i$, we get

$$\nabla \xi = \sum_{i=1}^{4} \frac{e_i}{\sqrt{g_i}} \frac{\partial}{\partial x^i} H_i \xi + \frac{1}{4} \sum_{i=1}^{4} \sum_{k \neq i} \frac{e_k}{g_i \sqrt{g_k}} \frac{\partial g_i}{\partial x^k} H_k \xi.$$

In the double sum we may sum over k from 1 to 4, and over the same values of i, except the value i = k:

$$\nabla \xi = \sum_{k=1}^{4} \frac{e_k}{\sqrt{g_k}} \frac{\partial}{\partial x^k} H_k \xi + \frac{1}{4} \sum_{k=1}^{4} \sum_{i \neq k} \frac{e_k}{g_i \sqrt{g_k}} \frac{\partial g_i}{\partial x^k} H_k \xi. \quad (12)$$

Thus we can set

$$\nabla = \sum_{k=1}^{4} e_k H_k \widetilde{D}_k,$$

*It is also easy to get from Eq. (2) the rule for differentiation of the product of a matrix A and a spinor ξ: $D_i A \xi = (D_i A)\xi + A D_i \xi$. When we use Eq. (10) to differentiate a vector expressed as a bilinear combination of the components of a spinor, we get the correct result on the basis of the remark just made. If, on the other hand, we take the requirement that the rule for differentiation of a vector shall hold as the basis for finding $D_i \xi$, the result obtained is not only incorrect, but also ambiguous. But if we follow the Cartan procedure there are, as we see, no ambiguities.

where, according to Eq. (12),

$$\widetilde{D}_k = \frac{1}{\sqrt{g_k}}\left(\frac{\partial}{\partial x^k} + \frac{1}{4}\sum_{l \neq k}\frac{1}{g_l}\frac{\partial g_l}{\partial x^k}\right). \tag{13}$$

The operator $-i\hbar\widetilde{D}_k$ is also taken for the momentum P_k.[1] It has, however, nothing in common with the covariant differentiation operator D_k, as can be verified easily by particular examples. The correctness of the results obtained by the use of Eq. (13) is explained by the fact that

$$\sum_{k=1}^{4} e_k H_k D_k \xi = \sum_{k=1}^{4} e_k H_k \widetilde{D}_k \xi.$$

If, however, one uses the operators $\widetilde{D}_k$ not in the linear combination with the matrices H_k, but separately, then they naturally lead to incorrect results.

Let us consider, for example, the cylindrical coordinates ρ, φ, z, x^4, for which

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2 - (dx^4)^2,$$

i.e.,

$$g_1 = g_3 = g_4 = 1, \quad g_2 = \rho^2.$$

From Eqs. (10) and (13) we find

$$D_1 = \partial/\partial\rho, \quad D_2 = (\partial/\partial\varphi + {}'H_2H_1/2)/\rho \text{ etc.}$$
$$\widetilde{D}_1 = \partial/\partial\rho + 1/2\rho, \quad \widetilde{D}_2 = \partial/\partial\varphi \text{ etc.}$$

Let ξ be a constant spinor, prescribed in terms of its Cartesian components. In cylindrical coordinates the column representing the same spinor field is[1]

$$\xi' = \left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2} H_1H_2\right)\xi.$$

Since the field is constant, we must have $D_1\xi' = 0$, $D_2\xi' = 0$, and these equations are indeed verified. But $\widetilde{D}_1\xi' \neq 0$, $\widetilde{D}_2\xi' \neq 0$, so that the operators D_1, D_2 cannot possibly be called covariant derivatives.

We note that the expression (10) can also be obtained by starting from the requirement that the covariant derivative of a constant spinor field must be zero.

3. THE ANGULAR-MOMENTUM OPERATOR

The components of the operator for the orbital angular momentum are defined by the equations

$$K_1 = -i\hbar(R_2D_3 - R_3D_2) \text{ etc.}$$

where R_k is a Galilean covariant component of the radius vector. The complete (three-dimensional) operator for the angular momentum is

$$K = K_1H_1 + K_2H_2 + K_3H_3.$$

In spherical coordinates θ, φ, r we get

$$K = \frac{\hbar}{i}\left\{\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right)H_2 - \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}H_1 - H_1H_2H_3\right\}, \tag{14}$$

whereas the use of the operators $\widetilde{D}_k$ instead of D_k would give

$$\widetilde{K} = \frac{\hbar}{i}\left\{\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right)H_2 - \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}H_1\right\}. \tag{15}$$

The eigenvalues of the operator (15) are all positive integers, excluding zero,[1] whereas in the nonrelativistic theory the angular momentum can also have the value zero. The operator (14), on the other hand, admits also the zero eigenvalue, because the eigenvalues of the matrix $iH_1H_2H_3$ are ± 1 and this matrix commutes with $\widetilde{K}$.

There is, however, also a third possibility. Let us define the angular momentum as the (three-dimensional) bivector with the components

$$K_{mn} = R_mP_n - R_nP_m.$$

According to Eq. (4) the Cartan matrix of this bivector is

$$\hat{K} = H_2H_3K_{23} + H_3H_1K_{31} + H_1H_2K_{12}. \tag{16}$$

The operator (16) is antihermitian, and therefore we must multiply it by i. In spherical coordinates we get

$$\hat{\widetilde{K}} = -\hbar\left\{\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right)H_1H_3 + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}H_2H_3\right\}. \tag{17}$$

The eigenvalues and eigenfunctions of the operators (17) and (15) are the same (apart from interchanges of components), but from the point of view of the representation of operators by Cartan matrices the expression (17) must be given preference.

The results obtained in this paper are also correct in the presence of gravitational fields, provided the coordinate bases remain orthogonal, as, for example, in the case of the Schwartzschild line element.[3]

[1] V. A. Fock, Начала квантовой механики (Principles of Quantum Mechanics), KUBUCh, 1932.

[2] A. A. Sokolov and D. D. Ivanenko, Квантовая теория поля (Quantum Field Theory), Gostekhizdat, 1952.

[3] É. Cartan, Théorie des Spineurs, Hermann, Paris, 1938; Russian Transl., IIL, 1947.

[4] N. E. Kochin, Векторное исчисление и начала тензорного исчисления (Vector Calculus and Principles of Tensor Calculus), GONTI, 1938.

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