ON THE SOLUTION OF THE KINETIC EQUATION FOR A PLASMA IN A VARIABLE MAGNETIC FIELD

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The motion of a completely ionized plasma (collisions being neglected) along a narrow magnetic tube of an axially symmetrical magnetic field is considered by means of the kinetic equation. The equation is solved under the assumption of sufficiently slow variation of the magnetic field. Canonical variables are chosen as the independent variables of the distribution function.

We consider a completely ionized plasma without collisions, contained in a narrow magnetic tube of an axially magnetic field (the axis of the tube coincides with the axis of symmetry of the field). The motion of the charge along the tube is described by the following equations[1] (in the Hamiltonian form)

$$\dot{V}_{\parallel} = -\partial\mathcal{H}/\partial x, \qquad \dot{x} = \partial\mathcal{H}/\partial V_{\parallel},$$
$$\mathcal{H}(J_{\perp}, V_{\parallel}, x, t) = {}^1/_2\,[V_{\parallel}^2 + J_{\perp}H(x, \mu)],$$
$$J_{\perp} \equiv V_{\perp}^2/H(x,\mu) = \text{const.} \tag{1}$$

Here $V_{\parallel}$ and $V_{\perp}$ are the components of the velocity of the charge along and perpendicular to the magnetic tube, and μ is a time dependent parameter.

We carry out a canonical transformation of the variables x, $V_{\parallel}$ to the angle variable φ and the action variable $J_{\parallel}$:

$$V_{\parallel} = \partial S(J_{\perp}, J_{\parallel}, x, t)/\partial x, \qquad \varphi = \partial S(J_{\perp}, J_{\parallel}, x, t)/\partial J_{\parallel},$$
$$\mathcal{H}' = \mathcal{H}(J_{\perp}, J_{\parallel}, t) + \partial S(J_{\perp}, J_{\parallel}, x, t)/\partial t. \tag{2}$$

The generating function* S and the action variable $J_{\parallel}$ are defined by the equations

$$S = \int [2\mathcal{H} - J_{\perp}H(x, \mu)]^{1/2}\,dx,$$
$$J_{\parallel} = \frac{1}{2\pi}\oint [2\mathcal{H} - J_{\perp}H(x,\mu)]^{1/2}\,dx, \tag{3}$$

where the integration is carried out along the trajectory of the charge for a fixed value of the parameter μ. In the variables φ and $J_{\parallel}$, the system (1) takes the form

$$\dot{J}_{\parallel} = \partial h/\partial\varphi, \qquad \dot{\varphi} = \omega - \partial h/\partial J_{\parallel},$$
$$h \equiv -\partial S/\partial t, \qquad \omega \equiv \partial\mathcal{H}/\partial J_{\parallel}. \tag{4}$$

The system (4) corresponds to the Liouville equation for the distribution function* F

$$\frac{\partial F}{\partial t} + \frac{\partial h}{\partial\varphi}\frac{\partial F}{\partial J_{\parallel}} + \left(\omega - \frac{\partial h}{\partial J_{\parallel}}\right)\frac{\partial F}{\partial\varphi} = 0. \tag{5}$$

We assume that the magnetic field changes slowly with time

$$H^{-1}\partial H/\partial t \ll \langle V_{\perp}^2\rangle^{1/2}/a, \tag{6}$$

where a is a distance of the order of the size of the inhomogeneity of the magnetic field along the magnetic tube. In this case we can look for a solution of Eq. (5) in the form of an expansion in powers of the small quantity $1/\omega$:

$$F = F^{(0)} + \lambda^{-1}F^{(1)} + \ldots,$$
$$\langle F^{(1)}\rangle_{\varphi} = \ldots = 0, \qquad \omega \equiv \lambda\omega_1, \tag{7}$$

where the index φ next to the averaging symbol denotes averaging over the angle variable φ. Substitution of (7) in (5) gives

$$\frac{\partial F^{(0)}}{\partial\varphi} = 0, \qquad \frac{\partial F^{(0)}}{\partial t} = 0,$$
$$F^{(1)} = -\frac{1}{\omega_1}\frac{\partial F^{(0)}}{\partial J_{\parallel}}(h - \langle h\rangle_{\varphi}). \tag{8}$$

Thus, limiting ourselves to terms proportional to

*Generally speaking, in the case of an arbitrary dependence of μ on the time, the generating function is equal to $S + S_1(J_{\perp}, J_{\parallel}, \mu)$, where S_1 is a certain function (see reference 2). However, in our approximation, S_1 can be omitted.

*As is explained below, in our case no sufficiently slow change of the magnetic field of the space charge is produced by the motion of the plasma. Therefore there is no necessity of making a distinction between the distribution functions of the electr ons and ions.

$1/\omega$, we obtain the following solution of Eq. (5):

$$F(J_\perp, J_\parallel, \varphi, t) = F^{(0)}(J_\perp, J_\parallel) - \frac{1}{\omega}\frac{\partial F^{(0)}}{\partial J_\parallel}(h - \langle h \rangle_\varphi). \quad (9)$$

We shall give another derivation of Eq. (9). For this purpose we note that under conditions (6), the system (4) is a system with a rapidly changing phase.[3] Its approximate solution has the form

$$J_\parallel = \bar{J}_\parallel + (h - \langle h \rangle_\varphi)/\omega, \qquad d\bar{J}_\parallel/dt = 0. \quad (10)$$

If at the initial moment the distribution function is equal to $F^{(0)}(J_\perp, J_\parallel)$, then at the instant t we have

$$F(J_\perp, J_\parallel, \varphi, t) = F^{(0)}(J_\perp, \bar{J}_\parallel) \approx F^{(0)}(J_\perp, J_\parallel) - \omega^{-1}(\partial F^{(0)}/\partial J_\parallel)(h - \langle h \rangle_\varphi). \quad (11)$$

According to (10), the action variable itself, $J_\parallel$ is not adiabatically invariant, but its average value $\bar{J}_\parallel$ over the period of the unperturbed motion of the system (4) is (i.e., for μ = const.). As is evident from (11), the function $F^{(0)}$ in (9) can be set equal to the distribution function at the initial instant.

Let us consider a specific example. We describe the magnetic field in the form

$$H(x, t) = \bar{H}(t)(1 + x^2/a^2), \quad (12)$$

and the distribution function at the initial instant as a Maxwell distribution

$$F^{(0)} = N_0(\pi\Theta)^{-3/2}\exp[-(V_\perp^2 + V_\parallel^2)/\Theta]. \quad (13)$$

According to Eq. (9), we get

$$F(x, V_\parallel, t) = N_0(\pi\Theta)^{-3/2}\left[1 - \frac{1}{2\Theta}\frac{\dot{\bar{H}}}{\bar{H}}\left(\frac{\bar{H}_0}{\bar{H}}\right)^{1/2} V_\parallel x\right] \times \exp\left\{-\frac{1}{\Theta}\left(\frac{\bar{H}_0}{\bar{H}}\right)^{1/2}\left[\left(\frac{\bar{H}_0}{\bar{H}}\right)^{1/2}\frac{V_\perp^2}{\Phi(x, t)} + V_\parallel^2\right]\right\}, \quad (14)$$

where

$$\Phi(x, t) = (a^2 + x^2)/[a^2 + (\bar{H}/\bar{H}_0)^{1/2}x^2]. \quad (15)$$

Calculations of the moments from zeroth to third order in the velocity relative to (14) leads to the following relations:

$$\frac{n}{n_0} \equiv \frac{N}{N_0}\frac{\bar{H}_0}{\bar{H}} = \left(\frac{\bar{H}}{\bar{H}_0}\right)^{1/4}\Phi = \begin{cases} (\bar{H}/\bar{H}_0)^{1/4} & \text{for } x \to 0, \\ (\bar{H}/\bar{H}_0)^{-1/4} & \text{for } x \to \pm\infty, \end{cases}$$

$$\langle V_\perp^2\rangle/\langle V_\perp^2\rangle_0 = (\bar{H}/\bar{H}_0)\Phi, \qquad \langle V_\parallel^2\rangle/\langle V_\parallel^2\rangle_0 = (\bar{H}/\bar{H}_0)^{1/2},$$

$$\langle V_\perp^2\rangle/2\langle V_\parallel^2\rangle = (\bar{H}/\bar{H}_0)^{1/2}\Phi \geqslant 1 \qquad \langle V_\parallel\rangle = -(\dot{\bar{H}}/4\bar{H})x,$$

$$\langle V_\perp^2 V_\parallel\rangle = \langle V_\perp^2\rangle\langle V_\parallel\rangle, \qquad \langle V_\parallel^3\rangle = 3\langle V_\parallel^2\rangle\langle V_\parallel\rangle. \quad (16)$$

According to (16), as the magnetic field is increased the plasma is compressed to the region of its minimum value. The kinetic temperatures of the plasma increase. In this case the velocity distribution becomes anisotropic: the transverse temperature is greater than the longitudinal. The heat flow is equal to 0 and the compression of the plasma is adiabatic.

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[1] H. Alfvén, Cosmical Electrodynamics, Oxford, 1950 (Russian translation, IIL, 1952).

[2] V. N. Volosov, Dokl. Akad. Nauk SSSR **123**, 587 (1958).

[3] N. N. Bogolyubov, and Yu. A. Mitropol'skiĭ, Асимртотические методы в теории нелинейных колебаний (Asymptotic Methods in the Theory of Nonlinear Oscillations) Fizmatizdat, 2nd ed., 1958.

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