γ_c as compared with the values obtained by the method of the Rome group[1] (column 2).

With the exception of one shower,[5] the inelasticity factor is substantially smaller than 1. The error in the determination of γ_c and K, arising out of the fluctuation of the angular distribution of particles can be taken into account analogously to the procedure in reference 9.

The average value of the transverse momentum (column 7) lies within the limits of 1 to 2. To explain how the assumption of constant transverse momentum influences the estimated values of γ_c and K, the latter were calculated from Eqs. (4) and (5) for the value $p_{\perp i} \approx 1$. The assumption of constant transverse momentum ($p_\perp \approx 1$) does not lead to substantial changes of the estimated values (columns 8 to 11). This makes it possible to generalize the described method for an estimate of the energy characteristics (γ_c and K) in showers in which only the angular distribution of secondary shower particles is known. It should be noted that the estimated values of γ_c found by such a method are in good agreement with the values obtained by Takibaev under the assumption of a power-law energy spectrum of produced mesons.

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ON THE METHODS OF BORN AND PAIS FOR FINDING PHASE SHIFTS

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As is well known, in cases in which the Born approximation for finding the phase shifts is not applicable, we must use some other more accurate method, for example the method of Pais. The purpose of this note is to give a brief derivation of the approximations of Born[1] and Pais[2] and also a numerical comparison of the phase shifts obtained by these two methods.

Let us write the Schrödinger equation in the following form:

$$y'' + [k^2 - U - l(l+1)/r^2]\, y = 0, \tag{1a}$$

$$u'' + [k^2 - l(l+1)/r^2]\, u = 0, \tag{1b}$$

$$v'' + [k^2 - (l(l+1) - a^2)/r^2]\, v = 0, \tag{1c}$$

$k = mv/\hbar$, and a^2 is a certain constant. The interaction potential is connected with U by the relation $U = (2m/\hbar^2)V$. The exact solutions of Eqs. (1b) and (1c) are well-known:

$$u = \sqrt{\pi kr/2}\, J_{l+1/2}(kr), \qquad v = \sqrt{\pi kr/2}\, J_{\sqrt{(l+1/2)^2 - a^2}}(kr). \tag{2}$$

They satisfy the following boundary conditions:

$$u(0) = 0, \quad u(\infty) \to \sin(kr - l\pi/2),$$

$$v(0) = 0, \quad v(\infty) \to \sin\left(kr - \frac{\pi}{2}\sqrt{(l+1/2)^2 - a^2} + \frac{\pi}{4}\right). \tag{3}$$

The exact solution of Eq. (1a) satisfies the boundary conditions

$$y(0) = 0, \quad y(\infty) \to \sin(kr - l\pi/2 + \eta_l). \tag{4}$$

If we now require that the solution v of Eq. (1c) satisfy the same boundary conditions (4) as the exact solution of Eq. (1a), then the constant a^2 must be

$$a^2 = 4\pi^{-2}[-\eta_l^2 + \pi(l + \tfrac{1}{2})\eta_l]. \tag{5}$$

Then

$$v = \sqrt{\pi kr/2}\, J_{(l+1/2) - 2\eta_l/\pi}(kr). \tag{6}$$

Let us multiply Eq. (1a) by u and Eq. (1b) by y, subtract one equation from the other, and integrate from zero to ∞, taking account of the boundary conditions (3) and (4) for u and y. Furthermore, in the integral that contains the interaction potential V, the solution u is inserted instead of the exact solution y. This procedure gives the well-known Born approximation:

$$\eta_l = -\frac{\pi m}{\hbar^2}\int_0^\infty V(r) J^2_{l+1/2}(kr)\, r\, dr. \qquad (7)$$

Applying this same procedure to Eqs. (1b) and (1c), we get the well-known Kapteyn integral[3]

$$\int_0^\infty J_\mu(\alpha t) J_\nu(\alpha t)\frac{dt}{t} = \begin{cases} \dfrac{2}{\pi}\dfrac{\sin[\pi(\nu-\mu)/2]}{\nu^2-\mu^2}, & \mu \neq \nu \\ \dfrac{1}{2\mu}, & \mu = \nu. \end{cases} \qquad (8)$$

Finally, applying this procedure to Eqs. (1a) and (1c), where the solution for v is given by Eq. (6), and using Eq. (8), we get the Pais approximation for the phase shifts:

$$\frac{2l+1-2\eta_l/\pi}{2l+1-4\eta_l/\pi}\eta_l = -\frac{\pi m}{\hbar^2}\int_0^\infty V(r) J^2_{l+1/2-2\eta_l/\pi}(kr)\, r\, dr. \qquad (9)$$

Pais obtained this formula by means of a variational principle. We see that one can obtain the approximate formulas of Born and Pais for the phase shifts by using a single kind of procedure.

The examples considered below show the accuracies of the Born and Pais approximations (it must be emphasized that the Pais formula (9) is incorrect for the zeroth-order phase shift). For the Gauss potential $V(r) = -V_0 \exp(-\alpha^2 r^2)$, Eqs. (7) and (9) give

$$\eta_{l\,\mathrm{Born}} = \eta_l^{(1)} = \frac{\pi M V_0}{4\hbar^2\alpha^2}\exp\left(-\frac{k^2}{2\alpha^2}\right) J_{l+1/2}\left(\frac{k^2}{2\alpha^2}\right),$$
$$\frac{2l+1-2\eta_l/\pi}{2l+1-4\eta_l/\pi}\eta_l = \frac{\pi M V_0}{4\hbar^2\alpha^2}\exp\left(-\frac{k^2}{2\alpha^2}\right) J_{l+1/2-2\eta_l/\pi}\left(\frac{k^2}{2\alpha^2}\right). \qquad (10)$$

Considering the scattering of a neutron by a proton (M is the mass of the proton) at 100 Mev, and choosing for the constants the values $V_0 = 45$ Mev and $\alpha^2 = 0.266 \times 10^{26}\ \mathrm{cm}^{-2}$, we get the following values of the phase shifts by the Pais method: $\eta_1 = 0.534$, $\eta_2 = 0.221$, whereas the Born method gives $\eta_1 = 0.487$ and $\eta_2 = 0.197$. Since the second Born approximation gives better results than the first, we shall compare the values obtained above for the phase shifts with the results of the second Born approximation,[4] $\eta_1 = 0.552$ and $\eta_2 = 0.213$. We see that the Pais method gives considerably better results than the first Born approximation.

For large values of l the phase shifts η_l calculated by the Born and Pais methods approach each other, as can be seen from Eqs. (7) and (9).

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THE RADIATIVE CORRECTION TO THE MASS OF THE ELECTRON IN NONLINEAR ELECTRODYNAMICS

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It has been shown in a previous paper[1] that if in analogy with Einstein's theory of the gravitational field one describes the electromagnetic (vector) field as the curvature of an auxiliary "space" with the metric $ds = \gamma_i dx^i$, then one can obtain a nonlinear Lagrangian of the electromagnetic field, which in the case of a static spherically symmetrical field leads to the potential

$$\varphi = (e/r_0\sqrt{2})\sinh(r_0\sqrt{2}/r), \qquad r_0 = e^2/m_0c^2. \qquad (1)$$

This gives for the classical (unquantized) field mass of a stationary electron $m_{cl} \approx m_0/5$, where m_0 is the experimental rest mass of the electron (the value $m_{cl} \approx m_0/3$ is erroneously given in reference 1).

To calculate the radiative (quantum) correction Δm_q to the mass of the electron, caused by the interaction of the stationary electron with the photon and electron-positron backgrounds, we must first of all find the wave solution of the field equations corresponding to the nonlinear Lagrangian in question. Since this is practically unfeasible because of the great mathematical difficulties, it is not without interest to try to give at least a preliminary and approximate estimate of the size of Δm_q. The idea of the calculation is as follows.

In the nonlinear theory under consideration, one gets in accordance with Eq. (1) for the energy of a stationary charge e situated in the field of another charge e, not the value $E_1 = e\varphi_1 = e^2/r$, but instead