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PASSAGE OF PARTICLES THROUGH A PLASMA

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The Green's function method and the diagram technique are used to calculate the energy loss per unit time by a particle passing through a plasma. Numerical values of the factors in the argument of the logarithm have been obtained for limiting cases.

1. INTRODUCTION

CALCULATION of the decelerating ability of a plasma by the method of paired collisions leads to a logarithmic divergence connected with the long-range action of the Coulomb forces. The screening effect of the medium can be neglected only when the distance between the particles is less than the Debye radius. To calculate the contribution due to close-range collisions, the Coulomb field is cut off at the Debye radius. Akhiezer and Sitenko[1] used the kinetic equation with a self-consistent field to calculate the long-range collisions, at which the momentum transferred is much less than the reciprocal of the Debye radius. These methods do not lead to a correct description of the interaction of particles separated by a distance on the order of the Debye radius, and the results have therefore only logarithmic accuracy. In the present paper we express the decelerating ability in terms of a correlation function, which is a particular case of a two-particle Green's function. To calculate the latter we use the diagram technique, the convenience of which lies in the fact that it permits us to sum an infinite number of essential terms. Furthermore, by estimating the discarded graphs, we can readily determine the accuracy of the obtained results.

2. TRANSITION PROBABILITY

We consider a system of interacting particles which are in thermal equilibrium. The Hamiltonian of the system is

$$H = H_0 + H_1, \qquad H_0 = \sum \varepsilon_{\mathbf{p}} a_{\mathbf{p}}^+ a_{\mathbf{p}},$$

$$H_1 = \tfrac{1}{2}\sum V_{\mathbf{q}} a_{\mathbf{p}}^+ a_{\mathbf{p}'}^+ a_{\mathbf{p}'-\mathbf{q}} a_{\mathbf{p}+\mathbf{q}} \tag{1}$$

(we use a system of units in which $m = \hbar = e^2 = 1$), where $a_{\mathbf{p}}^+$ and $a_{\mathbf{p}}$ are the operators of production and annihilation of particles with momentum $\mathbf{p}$, $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2$, and $V_{\mathbf{q}}$ is the Fourier component of the interaction potential. For particles interacting in accordance with Coulomb's law, we have $V_{\mathbf{q}} = 4\pi/\mathbf{q}^2$. The particle passing through the medium has a mass M and a velocity v. The Hamiltonian of the interaction between the particle and the medium has the form

$$H_i = \sum V_{\mathbf{q}} a_{\mathbf{p}}^+ \alpha_{\mathbf{p}_1-\mathbf{q}}^+ \alpha_{\mathbf{p}_1} a_{\mathbf{p}-\mathbf{q}}, \tag{2}$$

where $\alpha_{\mathbf{p}}^+$ and $\alpha_{\mathbf{p}}$ are the operators of production and annihilation of the passing particle.

We consider the particle to be sufficiently fast, $e^2/\hbar v \ll 1$, so that its interaction with the particles of the medium can be treated by perturbation theory. The probability of a transition in which the particle goes from a state with momentum $\mathbf{p}_1 = M\mathbf{v}$ into a state with momentum $\mathbf{p}_1 - \mathbf{q}$, and the medium goes from state n into state m, is given by the known equation

$$w_{\mathbf{q}} = 2\pi |\langle m, \mathbf{p}_1 - \mathbf{q} | H_i | n, \mathbf{p}_1\rangle|^2 \times \delta(E_m - E_n - \varepsilon_{\mathbf{p}_1} + \varepsilon_{\mathbf{p}_1-\mathbf{q}}), \tag{3}$$

$$\langle m, \mathbf{p}_1 - \mathbf{q} | H_i | n, \mathbf{p}_1\rangle = V_{\mathbf{q}} \Big(\sum_{\mathbf{p}} a_{\mathbf{p}}^+ a_{\mathbf{p}-\mathbf{q}}\Big)_{mn}. \tag{4}$$

To obtain the total probability of a transition of the particle from a state with momentum $\mathbf{p}_1$ into a state with momentum $\mathbf{p}_1 - \mathbf{q}$ it is necessary to sum expression (3) over all final states of the system and to average over the initial ones with a density matrix

$$\rho = \exp\beta(\Omega + \mu N - H), \qquad \beta = 1/kT. \tag{5}$$

Using (4), we get

$$W_{\mathbf{q}} = 2\pi V_{\mathbf{q}}^2 \Phi_{\mathbf{q}}(\varepsilon_{\mathbf{p}_1} - \varepsilon_{\mathbf{p}_1-\mathbf{q}}), \tag{6}$$

where

$$\Phi_{\mathbf{q}}(\omega) = \sum_{mn} \Big| \Big(\sum_{\mathbf{p}} a_{\mathbf{p}}^+ a_{\mathbf{p}-\mathbf{q}}\Big)_{mn} \Big|^2 e^{\beta(\Omega + \mu N_n - E_n)} \times \delta(E_m - E_n - \omega). \tag{7}$$

3. TWO-PARTICLE GREEN'S FUNCTION

To find $\Phi_{\mathbf{q}}(\omega)$ let us determine its connection with the two-particle Green's function

$$K(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) = -i\,\mathrm{Sp}\exp[\beta(\Omega+\mu N-H)] \times T\{\psi^+(\mathbf{r}_1, t_1)\psi(\mathbf{r}_1, t_1)\psi^+(\mathbf{r}_2, t_2)\psi(\mathbf{r}_2, t_2)\}, \quad (8)$$

where

$$\psi(\mathbf{r}, t) = e^{-iHt}\sum_{\mathbf{p}} a_{\mathbf{p}} e^{i\mathbf{pr}} e^{iHt}, \qquad \psi^+(\mathbf{r}, t) = e^{-iHt}\sum_{\mathbf{p}} a^+_{\mathbf{p}} e^{-i\mathbf{pr}} e^{iHt}.$$

We are interested only in such a two-particle function, in which the coordinates and the times of the operators ψ and ψ^+ are pairwise equal. We can obtain for such a function dispersion relations analogous to those obtained by Landau[2] for single-particle Green's functions. It is easy to verify that K depends only on the differences $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $t = t_1 - t_2$. Let us make a Fourier transformation with respect to the variable $\mathbf{r}$

$$K_{\mathbf{q}}(t) = \int d\mathbf{r}e^{-i\mathbf{qr}}K(\mathbf{r}, t)$$

$$= \begin{cases} -i\sum_{mn} e^{\beta(\Omega+\mu N_n - E_n)} \left|\left(\sum_{\mathbf{p}} a^+_{\mathbf{p}} a_{\mathbf{p}-\mathbf{q}}\right)_{mn}\right|^2 e^{i(E_n-E_m)t}, & t>0, \\ -i\sum_{mn} e^{\beta(\Omega+\mu N_n - E_n)} \left|\left(\sum_{\mathbf{p}} a^+_{\mathbf{p}} a_{\mathbf{p}-\mathbf{q}}\right)_{nm}\right|^2 e^{i(E_m-E_n)t}, & t<0. \end{cases}$$

We interchange the summation indices in the lower sum. Then, using the definition (7) of the function $\Phi_{\mathbf{q}}(\omega)$, we get

$$K_{\mathbf{q}}(t) = \begin{cases} -i\int_{-\infty}^{\infty} d\omega\Phi_{\mathbf{q}}(\omega)\, e^{-i\omega t}, & t>0, \\ -i\int_{-\infty}^{\infty} d\omega\Phi_{\mathbf{q}}(\omega)\, e^{-\beta\omega}\, e^{-i\omega t}, & t<0. \end{cases}$$

We now go over to Fourier components in terms of the variable t

$$K(\mathbf{q}, \omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} K_{\mathbf{q}}(t)$$

$$= \int_{-\infty}^{\infty} d\omega'\Phi_{\mathbf{q}}(\omega') \left\{(1-e^{-\beta\omega'})\,\mathrm{P}\frac{1}{\omega'-\omega} + i\pi(1+e^{-\beta\omega'})\,\delta(\omega'-\omega)\right\}. \quad (9)$$

The symbol P denotes that the integral is taken in the sense of its principal value. Instead of $K(\mathbf{q}, \omega)$ it is more convenient to consider the function $\widetilde{K}(\mathbf{q}, \omega)$, which is analytic in the upper half plane of the variable ω:

$$\widetilde{K}(\mathbf{q}, \omega) = \int_{-\infty}^{\infty} d\omega'\Phi_{\mathbf{q}}(\omega')\,\frac{1-e^{-\beta\omega'}}{\omega'-\omega-i\delta}. \quad (10)$$

Equations (9) and (10) can be used to calculate $\Phi_{\mathbf{q}}(\omega)$:

$$\Phi_{\mathbf{q}}(\omega) = \frac{\mathrm{Im}\,K(\mathbf{q}, \omega)}{\pi[1+\exp(-\beta\omega)]} = \frac{\mathrm{Im}\,\widetilde{K}(q, \omega)}{\pi[1-\exp(-\beta\omega)]}. \quad (11)$$

To calculate $\widetilde{K}$ we use the technique employed by Abrikosov, Gor'kov, and Dzyaloshinskiĭ[3] to find single-particle functions. For this purpose we consider the function

$$\mathscr{K}(\mathbf{r}_1, \tau_1, \mathbf{r}_2, \tau_2) = \mathrm{Sp}\,e^{\beta(\Omega+\mu N-H)}\,T \times\{\bar{\psi}(\mathbf{r}_1, \tau_1)\psi(\mathbf{r}_1, \tau_1)\bar{\psi}(\mathbf{r}_2, \tau_2)\psi(\mathbf{r}_2, \tau_2)\}, \quad (12)$$

$$\psi(\mathbf{r}, \tau) = e^{-(\mu N-H)\tau}\sum_{\mathbf{p}} a_{\mathbf{p}} e^{i\mathbf{pr}} e^{(\mu N-H)\tau},$$

$$\bar{\psi}(\mathbf{r}, \tau) = e^{-(\mu N-H)\tau}\sum_{\mathbf{p}} a^+_{\mathbf{p}} e^{i\mathbf{pr}} e^{(\mu N-H)\tau}.$$

The value of $\mathscr{K}$, like that of K, depends only on the differences $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\tau = \tau_1 - \tau_2$; therefore, after transformations analogous to those made with the function K(t), we get

$$\mathscr{K}(\mathbf{q}, \tau) = \begin{cases} \int_{-\infty}^{\infty} d\omega\Phi_{\mathbf{q}}(\omega)\, e^{-\omega\tau}, & \tau>0, \\ \int_{-\infty}^{\infty} d\omega\Phi_{\mathbf{q}}(\omega)\, e^{-\omega(\beta+\tau)}, & \tau<0. \end{cases} \quad (13)$$

We shall detail below a method of calculating the function $\mathscr{K}$ in the interval $-\beta < \tau < \beta$. We shall show how, knowing $\mathscr{K}$ in this region, we can find the function $\Phi_{\mathbf{q}}(\omega)$. We shall assume that $\mathscr{K}(\tau)$ is given by (13) in the intervals $0 < \tau < \beta$, and is continued periodically outside this interval. Then the coefficients of its Fourier expansion are determined from

$$\mathscr{K}_n = \int_0^{\beta} \mathscr{K}(\mathbf{q}, \tau)\exp(2\pi i n\tau/\beta)\,d\tau.$$

Inserting the value of $\mathscr{K}(\mathbf{q}, \tau)$ from (13) and integrating with respect to τ we obtain

$$\mathscr{K}_n(\mathbf{q}) = \int_{-\infty}^{\infty} d\omega\Phi_{\mathbf{q}}(\omega)\frac{1-e^{-\beta\omega}}{\omega-2\pi i n/\beta}. \quad (14)$$

A direct comparison of (10) and (14) shows that

$$\mathscr{K}_n(\mathbf{q}) = \widetilde{K}(\mathbf{q}, 2\pi i n/\beta). \quad (15)$$

Equation (15) yields the function $\widetilde{K}(\mathbf{q}, \omega)$ for several values of the argument $\omega = 2\pi i n/\beta$. In addition, it is known that the function $\widetilde{K}(\mathbf{q}, \omega)$ is analytic in the upper half plane, and is therefore uniquely determined from its values at the indicated infinite set of points. Knowing the function $\widetilde{K}(\mathbf{q}, \omega)$ on the real axis, it is possible to find $\Phi_\mathbf{q}(\omega)$ from Eq. (11).

We can also calculate $K(\mathbf{q}, \tau)$ by the diagram technique. We introduce the following operators in the interaction representation

$$a_\mathbf{p}(\tau) = a_\mathbf{p} e^{-(\varepsilon_\mathbf{p}-\mu)\tau}, \quad \overset{+}{a}_\mathbf{p}(\tau) = \overset{+}{a}_\mathbf{p} e^{(\varepsilon_\mathbf{p}-\mu)\tau},$$

$$H_1(\tau) = e^{H_0\tau} H_1 e^{-H_0\tau}.$$

In Eq. (12) we go in the usual manner from the operators in the Heisenberg representation to operators in the interaction representation. In the region $0 < \tau_1 < \beta$, $0 < \tau_2 < \beta$ we obtain

$$\mathscr{K}(\mathbf{q}, \tau_1 - \tau_2) = \mathrm{Sp}\, e^{\beta(\Omega + \mu N - H_0)} \times T\Big\{ \sum_{\mathbf{p}_1, \mathbf{p}_2} \overset{+}{a}_{\mathbf{p}_1}(\tau_1)\, a_{\mathbf{p}_1 - \mathbf{q}}(\tau_1)\, \overset{+}{a}_{\mathbf{p}_2}(\tau_2)\, a_{\mathbf{p}_2 + \mathbf{q}}(\tau_2) \times \exp\Big[-\int_0^\beta H_1(\tau)\, d\tau\Big]\Big\}. \qquad (16)$$

The factor $\exp\left[-\int_0^\beta H_1(\tau)\, d\tau\right]$ can be expanded in powers of H_1. Inserting this expansion into (16), we obtain an expansion of $\mathscr{K}(\mathbf{q}, \tau)$ in powers of the interaction. Each term of the expansion can be represented by a diagram. Corresponding to each dotted line on the diagram is a factor $V_\mathbf{q}$, and to each solid line there corresponds a null Green's function of the particle

$$G(\mathbf{p}, \tau) = \begin{cases} (1 \mp n_\mathbf{p})\, e^{-(\varepsilon_\mathbf{p} - \mu)\tau}, & \tau > 0, \\ \mp n_\mathbf{p}\, e^{-(\varepsilon_\mathbf{p} - \mu)\tau}, & \tau < 0, \end{cases} \qquad (17)$$

$$n_\mathbf{p} = [e^{(\varepsilon_\mathbf{p} - \mu)\beta} \pm 1]^{-1}.$$

The upper sign pertains to Fermi particles and the lower one to Bose particles. The function $\mathscr{K}(\mathbf{q}, \tau)$ is represented by the sum of all the connected graphs that have two outer vertices. Two solid lines with momenta that differ by $\mathbf{q}$ meet at each of these vertices. Several first-order diagrams are shown in the figure.

Calculation of the probability of a transition by considering the interaction of the particle with the medium as paired collisions between the passing particle and the particles of the medium is equivalent to taking into account only the simplest diagram a in calculating the function $\mathscr{K}$. In the case of a plasma, such a method gives for the decelerating ability an expression that diverges logarithmically at small $\mathbf{q}$. To eliminate this divergence it is necessary to take into account the screening action of the medium, i.e., more complicated graphs. Terms expressed by the graphs b or d contain factors $V_\mathbf{q} = 4\pi/\mathbf{q}^2$ and therefore become comparable at small $\mathbf{q}$ with the term represented by graph a. Let us segregate all the graphs that contain $V_\mathbf{q}$; for this purpose we denote by $\Pi(\mathbf{q}, \tau)$ the set of the compact parts of the function $\mathscr{K}$, i.e., those containing no parts that are connected only by one dotted line. Examples of such graphs are a, c, f, and g. The function $\mathscr{K}(\mathbf{q}, \tau)$ is related to $\Pi(\mathbf{q}, \tau)$ by the integral equation

$$\mathscr{K}(\mathbf{q}, \tau) = \Pi(\mathbf{q}, \tau) + \int_0^\beta d\tau'\, \Pi(\mathbf{q}, \tau - \tau')\, V_\mathbf{q}\, \mathscr{K}(\mathbf{q}, \tau'). \qquad (18)$$

Expanding in a Fourier series in τ and solving the resultant algebraic equation, we find

$$\mathscr{K}_n(\mathbf{q}) = \frac{\Pi_n(\mathbf{q})}{1 - V_\mathbf{q}\Pi_n(\mathbf{q})},$$

$$\text{where } \Pi_n(\mathbf{q}) = \int_0^\beta \Pi(\mathbf{q}, \tau) \exp\left(\frac{2\pi i n \tau}{\beta}\right) d\tau.$$

Inserting the expression obtained for $K_n(\mathbf{q})$ in (15) we get

$$\widetilde{K}(\mathbf{q}, \omega) = \Pi(\mathbf{q}, \omega) / [1 - V_\mathbf{q}\Pi(\mathbf{q}, \omega)], \qquad (19)$$

where $\Pi(\mathbf{q}, \omega)$ is a function analytic in the upper half plane of the variable ω, which coincides with $\Pi_n(\mathbf{q})$ at the points $\omega = 2\pi i n/\beta$.

For the transition probability we obtain from Eqs. (6), (11), and (19)

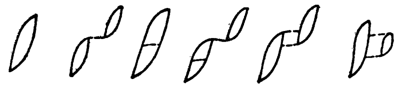
$$W_\mathbf{q} = \frac{2V_\mathbf{q}^2}{1 - \exp(-\beta\omega)} \mathrm{Im} \frac{\Pi(\mathbf{q}, \omega)}{1 - V_\mathbf{q}\Pi(\mathbf{q}, \omega)}, \qquad (20)$$

where $\omega = \epsilon_{\mathbf{p}_1} - \epsilon_{\mathbf{p}_1 - \mathbf{q}}$.

In the derivation of (20) we made no use of the properties of the medium, but the function $\Pi(\mathbf{q}, \omega)$ can be calculated only in limiting cases. For a sufficiently rarefied plasma and for a low-temperature electron gas of high density, it is possible to restrict the calculations to the term shown in graph a. From Eq. (17) we obtain in these cases by simple calculation

$$\Pi(\mathbf{q}, \omega) = -\int \frac{d^3p}{(2\pi)^3} \frac{n_{\mathbf{p}+\mathbf{q}/2} - n_{\mathbf{p}-\mathbf{q}/2}}{(\mathbf{p}\mathbf{q}) - \omega - i\delta}. \qquad (21)$$

For a rarefied plasma, the most important additions to Eq. (21) are represented by graphs of type f or g; their ratio to the principal term is



a b c d f g

on the order of $\sqrt{\beta^3 e^3 n}$. Thus, formula (21) is valid every time that the Debye theory is valid. At low temperatures the greatest correction is determined by the graph c; in this case the expansion parameter is the usual parameter of perturbation theory, $e^2 m/\hbar p_0$, where p_0 is the limiting momentum of the limiting momentum of the Fermi surface, connected with the particle density n by

$$n = p_0^3 / 3\pi\hbar^3. \quad (22)$$

4. PLASMA OSCILLATIONS

The spectrum and the damping of collective excitations that represent the density oscillations are determined by the poles of the function $\widetilde{K}(\mathbf{q}, \omega)$ in the lower half plane. From (19) we obtain an equation for the spectrum

$$1 = V_{\mathbf{q}} \Pi(\mathbf{q}, \omega). \quad (23)$$

Equation (23), with account of (21), is the same equation that Klimontovich and Silin[4] obtained with the aid of the quantum-kinetic equation. At small $\mathbf{q}$ we have

$$\Pi(\mathbf{q}, \omega) = n\mathbf{q}^2 / \omega^2. \quad (24)$$

Inserting this expression into (23), we obtain the Langmuir spectrum

$$\omega = \omega_0, \qquad \omega_0^2 = 4\pi n e^2 / m, \quad (25)$$

where the electron density is

$$n = (2\pi)^{-3} \int n_p d^3 p.$$

The damping of the excitations is determined by the imaginary part of ω, which can be found by calculating the imaginary part of $\Pi(\mathbf{q}, \omega)$. Calculation of $\mathrm{Im}\,\Pi(\mathbf{q}, \omega)$ from Eq. (21) leads to zero damping at zero temperature, and yields at high temperatures

$$\mathrm{Im}\,\omega = -\frac{(2\pi\beta)^{1/2}}{8\pi} \frac{\omega_0^4}{q^3} \exp\left(-\frac{\beta\omega_0^2}{2q^2}\right), \quad (26)$$

which agrees with the expression derived by Landau.[5] Taking account of higher-order terms in $\Pi(\mathbf{q}, \omega)$ which are represented by graphs of type f or g, it is possible to determine the damping due to the viscosity of the electron gas. It is present at zero temperature and drops more slowly with diminishing q at high temperature, although at $q^2 \sim \beta\omega_0^2$ this damping is much less than that calculated from Eq. (26).

5. DECELERATING ABILITY OF PLASMA

The energy lost by a passing particle per unit time is given by

$$dE / dt = (2\pi)^{-3} \int (\varepsilon_{p_1} - \varepsilon_{p_1 - \mathbf{q}}) W_{\mathbf{q}} d^3 q.$$

Inserting into this equation the value of $W_{\mathbf{q}}$ from (20) and $V_{\mathbf{q}} = 4\pi/q^2$ we get

$$-\frac{dE}{dt} = 8 \int_0^\infty dq \int_{-1}^{1} dx \frac{\omega}{1 - \exp(-\beta\omega)} \mathrm{Im} \frac{\Pi(q, \omega)}{q^2 - 4\pi\Pi(q, \omega)}, \quad (27)$$

where

$$x = \mathbf{vq}/vq, \qquad \omega = \varepsilon_{p_1} - \varepsilon_{p_1 - q} = vqx - q^2/2M,$$

v is the velocity and M the mass of the passing particle.

We consider the case when the particle moves with a velocity much greater than the mean thermal velocity of the electrons. The integral in (27) is broken up into two regions, $q > q_1$ and $q < q_1$, with q_1 chosen such that $1/\beta \gg q_1^2 \gg \kappa^2$, where

$$\varkappa^2 = -4\pi\Pi(q, 0) = 4\pi\beta n e^2 \quad (28)$$

(κ is the reciprocal of the Debye radius). In the first region we can neglect $4\pi\Pi(\mathbf{q}, \omega)$ compared with q^2, since Π is always of the order of κ^2. We calculate $\mathrm{Im}\,\Pi$ from Eq. (21)

$$\mathrm{Im}\,\Pi = n \frac{(2\pi\beta)^{1/2}}{2q} [1 - e^{-\beta\omega}] \exp\left\{-\frac{\beta}{2}\left(\frac{\omega}{q} - \frac{q}{2}\right)^2\right\}. \quad (29)$$

Inserting this expression into (27) we get

$$-\frac{dE}{dt}\bigg|_1 = 4n(2\pi\beta)^{1/2} \int_{q_1}^\infty dq \int_{-1}^{1} dx \frac{vx - q/2M}{q^2} \times \exp\left\{-\frac{\beta}{2}\left(vx - \frac{q}{2}\,\frac{M+1}{M}\right)\right\}. \quad (30)$$

Neglecting terms on the order $(v_e/v)^2$ (v_e is the mean thermal velocity of the electron), we have

$$-\frac{dE}{dt}\bigg|_1 = \frac{4\pi n}{v} \ln \frac{2Mv}{q_1(M+1)}.$$

In the integral in the second region, $q < q_1$, we can put $\omega = vqx$ and go from integration with respect to x to integration with respect to ω:

$$-\frac{dE}{dt}\bigg|_2 = 8 \int_0^{q_1} \frac{dq}{vq} \mathrm{Im} \int_{-vq}^{vq} \frac{\omega d\omega}{1 - \exp(-\beta\omega)} \frac{\Pi(q, \omega)}{q^2 - 4\pi\Pi(q, \omega)}. \quad (31)$$

In calculating the integral with respect to ω we make use of the fact that the integrand function differs from the function $\widetilde{K}(q, \omega)$ only by a multiplicative factor and is therefore analytic in the upper half-plane of the variable ω. We modify the contour integration in the upper half plane in such a way that the points $-vq$ and vq are joined not by the real axis but by a semicircle of radius vq with its center at the origin. On this contour $|\omega| = vq \gg v_e q$ and therefore, again neglecting terms of the order $(v_e/v)^2$, we use Eq. (24) for $\Pi(q, \omega)$. The integral with respect to ω in (31) becomes

$$\int \frac{d\omega}{1-\exp(-\beta\omega)} \frac{n\omega}{\omega^2-\omega_0^2}$$

The integrand is analytic in the upper half-plane, and we can therefore shift the contour back to the real axis, going around the poles $\omega = \pm\omega_0$ on the real axis from above. The imaginary part of this integral, which is of interest to us, is due to the detouring of these poles. When $vq < \omega_0$ it vanishes since the ends of the contour lie between the poles, and when $vq > \omega_0$ the imaginary part equals $\pi/2$. Inserting this into (31) we get

$$-(dE/dt)|_2 = 4\pi n \ln(vq_1/\omega_0).$$

Adding this expression to that obtained earlier for $(dE/dt)|_1$, we obtain an expression for the total losses:

$$-\frac{dE}{dt} = \frac{4\pi ne^4}{mv} \ln \frac{2Mm^{3/2}v^2}{\hbar(M+m)\sqrt{4\pi ne^2}} \qquad (32)$$

This equation was obtained by Akhiezer and Sitenko[1] accurate to within a logarithmic factor. The numerical factor given in Eq. (25) of that reference, however, contains an incorrect multiplier, 1.23, under the logarithm sign.

In this expression for the total losses, the component due to paired collisions can be separated from the component due to radiation of plasma waves only with logarithmic accuracy. Collisions in which the momentum transferred is much greater than the reciprocal Debye radius can be considered paired and their contribution is

$$-dE/dt = (4\pi ne^4/mv)\ln(mv/\hbar\varkappa).$$

When $\operatorname{Im}\Pi$ is small, the contribution to dE/dt produces in the integrand of Eq. (27) a pole located in the lower half plane near the real axis. These losses are due to radiation of long-lived plasma waves, the spectrum of which is determined by the pole of the integrand, i.e., by Eq. (23). Such waves can be considered excited as long as their damping, given by Eq. (26), is small compared with their frequency, i.e., when $q \ll \kappa$. Thus, the losses connected with their emission are

$$-dE/dt = (4\pi ne^4/mv)\ln(v/v_e).$$

Equation (32) shows that the total losses of a fast particle in a plasma are independent of the temperature. It can be verified that Eq. (32) is correct for any electron velocity distribution, provided the electrons can be considered free and their mean velocity is much less than the velocity of the passing particle. In particular, Eq. (32) is valid for energy losses in a high-density electron gas at zero temperature. It is interesting to note that since, in the first order of magnitude relative to $e^2/\hbar v_e$, there is no damping of the plasma waves in this case, it is possible to determine the losses connected with the radiation of the plasma waves not with logarithmic accuracy, but accurate to terms of order $e^2/\hbar v_e$. These losses are determined by the contribution of the pole, located on the real axis, of the integrand of Eq. (27). We must go around this pole from above, and therefore

$$-\frac{dE}{dt}\Big|_{pl} = 8\pi \int_0^\infty dq \frac{\omega dx}{1-\exp(-\beta\omega)} \Pi(q,\omega)\,\delta(q^2 - 4\pi\Pi(q,\omega)).$$

A substantial contribution to this integral is made by the region of small $q \sim \sqrt{p_0} \ll p_0$. It is therefore possible to assume $\omega = vqx$ and to replace the difference in the numerator of Eq. (21) for $\Pi(\mathbf{q}, vqx)$ by the derivative, $n_{\mathbf{p}+\mathbf{q}/2} - n_{\mathbf{p}-\mathbf{q}/2} = (\partial n_{\mathbf{p}}/\partial p)\mathbf{pq}/p$. In this case Π is independent of q and is a function of x:

$$\Pi = \frac{p_0}{\pi^2}\left(1 - \frac{vx}{p_0}\ln\frac{vx+p_0}{vx-p_0}\right).$$

It is necessary to integrate with respect to x in the limits from p_0/v to 1, for when $x < p_0/v$ the imaginary part of Π differs from zero. The integration is elementary and when $v \gg v_e$ we get

$$-\frac{dE}{dt}\Big|_{pl} = \frac{4\pi ne^4}{mv}\left(\ln\frac{mv}{2p_0} + \frac{4}{3}\right). \qquad (33)$$

If the decelerating particle is an electron, the numerical factor in the argument of the logarithm in (32) should be modified because of the influence of the exchange effect. Formally, this effect manifests itself in the fact that now the interaction Hamiltonian (2) becomes

$$H_i = \frac{1}{2}\sum_{\mathbf{p}p_1\mathbf{q}} V_{\mathbf{q}} a^+_{\mathbf{p}} a^+_{\mathbf{p}_1-\mathbf{q}} a_{p_1} a_{p-\mathbf{q}}$$

and an additional term $\Sigma_{\mathbf{p}} V_{\mathbf{p}-\mathbf{p}'} a^+_{\mathbf{p}} a_{\mathbf{p}-\mathbf{q}}$ appears in the matrix element (4). In addition, an electron that has a large energy after collision should be considered as primary. As is known, allowance for the exchange effect leads to an additional factor $\sqrt{e/8}$ in the argument of the logarithm. Instead of Eq. (32) we obtain the following equation for the slowing down of the electron

$$-\frac{dE}{dt}=\frac{4\pi ne^4}{mv}\left\{\ln\frac{m^{3/2}v^2}{4\hbar\sqrt{2\pi ne^2}}+1\right\}. \qquad (34)$$

Equation (32) can be obtained in the usual manner, by considering collisions with large momentum transfer as paired collisions of particles, and expressing the contribution of transitions with small momentum transfer to the decelerating ability in terms of the dielectric constant of the plasma $\epsilon(\omega) = 1 - (\omega_0/\omega)^2$.

Equation (27) can be used to find the energy lost by particles that do not have too high a velocity. It is easy to obtain a correction of order $(v_e/v)^2$ for Eq. (31). For this purpose it is necessary to calculate the integral in Eq. (30) with accuracy to terms of order $(v_e/v)^2$. We can no longer use the limiting value (24) for $\Pi(q, \omega)$ in Eq. (31), but must put

$$\Pi(q,\omega)=(nq^2/\omega^2)(1+3q^2/\beta\omega^2). \qquad (35)$$

The integrand of (31), with the value of $\Pi(q, \omega)$ from Eq. (35), has a pole in the upper half-plane, and this must be taken into account in evaluating the integral. The final formula for the losses has the form

$$-\frac{dE}{dt}=\frac{4\pi ne^4}{mv}\left\{\ln\frac{2Mmv^2}{(M+m)\hbar\omega_0}-\frac{2M+3m}{Mm\beta v^2}\right\}. \qquad (36)$$

It is also interesting to examine another limiting case, when a heavy particle moves with a velocity much lower than the thermal velocity of the electrons, $v \ll v_e$, but greater than the mean velocity of the ions, $v \gg v_i$, and sufficiently large to satisfy the perturbation-theory criterion $e^2/\hbar v \ll 1$. In this case it is more convenient to rewrite Eq. (27) as

$$-\frac{dE}{dt}=8\int_0^\infty dq\int_{-1}^{1}dx\,\frac{\omega}{1-\exp(-\beta\omega)}\,\frac{q^2\,\mathrm{Im}\,\Pi}{|q^2-4\pi\Pi|^2}. \qquad (37)$$

We express $\mathrm{Im}\,\Pi$ in terms of (29), allowing for contributions from both electrons and ions. The expression for $\Pi(q, \omega)$ contained in the denominator can be replaced, neglecting terms of order $(v/v_e)^2$, by $\Pi(q, 0)$, which is expressed in terms of the Debye radius with the aid of Eq. (28). It is necessary to account here only for the electron loops. Allowance for the ion loop leads to a correction on the order of $(v_i/v)^2$. Equation (37) becomes

$$-\frac{dE}{dt}=4(2\pi\beta)^{1/2}\int_0^\infty dq\,\frac{q^2}{(q^2+\varkappa^2)^2}\int_{-1}^{1}dx\left(vx-\frac{q}{2M}\right)$$
$$\times\left\{n_e\left[-\frac{\beta}{2m}\left(vx-\frac{q}{2m}\right)^2\right]\right.$$
$$\left.+\,n_i\exp\left[-\frac{\beta}{2M_i}\left(vx-q\frac{M+M_i}{2MM_i}\right)^2\right]\right\}. \qquad (38)$$

Equation (38) can be obtained by considering paired collisions of particles, interacting as $\exp(-\kappa r)/r$. This equality holds because the fast electrons have a chance to screen the field of the slower particle. The mean field of the particle moves together with the particle, without slowing down, and therefore in this limiting case the radiation of the plasma waves does not contribute to the energy loss.

Neglecting terms of order of $(v_i/v)^2$, we obtain for the losses in collisions with ions

$$-\frac{dE}{dt}\bigg|_i=\frac{4\pi n_i e^4}{M_i v}\left\{\ln\frac{MM_iv}{(M+M_i)\hbar\varkappa}-\frac{1}{2}\right\}. \qquad (39)$$

We calculate the losses due to collisions with electrons, neglecting terms of order $(v/v_e)^2$ and $(m/M)(v_e/v)^2$

$$-\frac{dE}{dt}\bigg|_e=\frac{2}{3}(2\pi\beta)^{1/2}\beta v^2 n_e e^4\{\ln(8m/\beta\hbar^2\varkappa)-C-1\}, \qquad (40)$$

where $C = 0.58$ is Euler's constant. Equations (39) and (40) contain the reciprocal Debye radius κ, which is determined from Eq. (28) and differs from the corresponding value used in thermodynamic functions in that it is expressed only in terms of the electron density, rather than the density of all charged particles.

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