ON CONVECTIVE MOTION OF A CONDUCTING FLUID BETWEEN PARALLEL VERTICAL PLATES IN A MAGNETIC FIELD

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Stationary convective motion of a conducting fluid between vertical parallel plates in a magnetic field is considered. An exact solution of the magnetohydrodynamic equations is obtained for the case of a constant vertical temperature gradient. The critical value of Grasshof's number is determined for the case when the temperature of both plates is the same.

1. The free convective motion of a conducting fluid between vertical plates maintained at a constant temperature in the presence of an external magnetic field was studied in detail by Gershuni and Zhukhovitskiĭ.[1,2] In this paper we shall generalize the results referring to stationary flow to the case when the temperature varies in the vertical direction. We shall also discuss the problem of the superposition of free and forced convection, thereby generalizing the well-known solution due to Hartmann.[3]

We consider the case of stationary convective motion of a fluid between vertical infinite plates $x = \pm\delta$, whose temperatures are respectively $T_-(z)$ and $T_+(z)$. We take the flow lines to be parallel to the plates, i.e., to the z axis. An external homogeneous transverse magnetic field $H_x = H_0$ is applied to the plates.

We write the general equations of magnetohydrodynamics:[4]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla\left(p + \frac{\mu H^2}{8\pi}\right) + \nu\nabla^2\mathbf{v} + \frac{\mu}{4\pi\rho}(\mathbf{H}\nabla)\mathbf{H} - \beta\mathbf{g}T, \quad (1.1)$$

$$\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{v}\nabla)\mathbf{H} = (\mathbf{H}\nabla)\mathbf{v} + \frac{c^2}{4\pi\sigma\mu}\nabla^2\mathbf{H}, \quad (1.2)$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = a\nabla^2 T, \quad (1.3)$$

$$\operatorname{div}\mathbf{v} = 0, \quad \operatorname{div}\mathbf{H} = 0, \quad (1.4)$$

where T is the temperature, p is the pressure, **v** and **H** are the velocity and field vectors, ρ, ν, μ, σ, a, β are, respectively, the density, the kinematic viscosity, the magnetic permeability, the conductivity, the thermal conductivity, and the coefficient of thermal expansion of the fluid. The problem under consideration has an exact solution of the form $v_x = v_y = 0$, $v_z = v(x)$, $H_x = H_x(x)$, $H_y = 0$, $H_z = H_z(x)$, $T = T(x, z)$, $p = p(x, z)$. If we take the positive direction of the z axis in the opposite direction to the force of gravity vector **g**, we obtain from Eqs. (1.1) — (1.4), the system of equations

$$0 = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(p + \frac{\mu H^2}{8\pi}\right) + \nu\frac{\partial^2 v}{\partial x^2} + \frac{\mu H_x}{4\pi\rho}\frac{\partial H_z}{\partial x} + \beta g T,$$

$$\frac{\partial}{\partial x}\left(p + \frac{\mu H^2}{8\pi}\right) = 0, \quad (1.5)$$

$$0 = H_x\frac{\partial v}{\partial x} + \frac{c^2}{4\pi\sigma\mu}\frac{\partial^2 H_z}{\partial x^2}, \quad (1.6)$$

$$v\frac{\partial T}{\partial z} = a\frac{\partial^2 T}{\partial x^2}, \quad (1.7)$$

where $H_x = \text{const}$, which follows from the equation $\operatorname{div}\mathbf{H} = 0$; $g = -g_z$.

Since the component of the field normal to the boundary must be continuous at $x = \pm\delta$, we have $H_x \equiv H_0$. On substituting T from (1.5) into (1.7) and on differentiating the resultant equation with respect to z we see that when $v \neq \text{const}$ we necessarily have $(\partial^2/\partial z^2)(p + \mu H^2/8\pi) = \text{const}$ and, consequently, $\partial T/\partial z = A = \text{const}$ (see reference 5). From this it follows that $T = Az + T_0(x)$. The expressions for the given plate temperatures may now be written in the form

$$T_-(z) = Az + T_0(+\delta), \quad T_+(z) = A_z + T_0(-\delta).$$

For the sake of definiteness we assume $T_-(z) < T_+(z)$. Then, on taking $T_0(0)$ as the reference point on the temperature scale, we have

$$T_-(z) = T(z, \delta) = Az - T_{w1},$$
$$T_+(z) = T(z, -\delta) = Az + T_{w2}, \quad (1.8)$$

where $T_{w_1} + T_{w_2}$ is the temperature difference between the plates which remains constant along the z axis. We shall subsequently use formulas (1.8) as the boundary conditions for the temperature.

Over the boundaries (at $x = \pm\delta$) the conditions of absence of slipping and of the continuity of the tangential component of the vector H must also be fulfilled, and reduce to the equalities

$$v(\pm\delta) = H_z(\pm\delta) = 0. \tag{1.9}$$

In addition we shall assume that the flow of fluid over the cross section of the gap between the plates is specified

$$Q = \int_{-\delta}^{\delta} v(x)\,dx. \tag{1.10}$$

If $Q = 0$, then Eq. (1.10) will be the condition that the lines of flow are closed, and the flow is free, since it will be due only to the force of buoyancy. However, if $Q \neq 0$ then the flow will be mixed, and forced convection will be superimposed on the free convection.

We now introduce the following dimensionless quantities

$$\xi = x/\delta, \quad \zeta = z/\delta, \quad u = v\delta/\nu, \quad h = H_z/H_0,$$
$$q = p\delta^2/\rho\nu^2, \quad \theta = T/A\delta = \zeta + \theta_0(\xi),$$
$$P = \nu/a, \quad P_m = 4\pi\sigma\nu/c^2, \quad M = (\mu H_0\delta/c)\sqrt{\sigma/\rho\nu},$$
$$G = \beta g A\delta^4/\nu^2, \tag{1.11}$$

where P and P_m are the ordinary and the magnetic Prandtl numbers, M is the Hartmann number, G is the Grasshof number. In terms of these new variables, the equations and the boundary conditions of our problem assume the following form

$$0 = -\frac{\partial q^*}{\partial\zeta} + \frac{\partial^2 u}{\partial\xi^2} + \frac{M^2}{P_m}\frac{\partial h}{\partial\xi} + G\theta, \quad \frac{\partial q^*}{\partial\xi} = 0, \tag{1.12}$$

$$0 = \frac{\partial u}{\partial\xi} + \frac{1}{P_m}\frac{\partial^2 h}{\partial\xi^2}, \tag{1.13}$$

$$u\frac{\partial\theta}{\partial\zeta} = \frac{1}{P}\Delta\theta, \tag{1.14}$$

$$u(\pm 1) = h(\pm 1) = 0, \quad Q^* = \int_{-1}^{1} u(\xi)\,d\xi,$$
$$\theta(\zeta, 1) = \zeta - \theta_{w1}, \quad \theta(\zeta, -1) = \zeta + \theta_{w2}, \tag{1.15}$$

where

$$q^* = q + M^2(h^2+1)/2P_m, \quad \theta_{wi} = T_{wi}/A\delta, \quad Q^* = Q/\nu.$$

Further analysis of our problems will be based on the system of equations (1.12) – (1.15).

2. In order to solve the system of equations (1.12) – (1.15) we substitute the expression $\theta = \zeta + \theta_0(\xi)$ into the first equation of (1.12) and into (1.14). We then obtain

$$0 = -\frac{\partial q^*}{\partial\zeta} + \frac{\partial^2 u}{\partial\xi^2} + \frac{M^2}{P_m}\frac{\partial h}{\partial\xi} + G\zeta + G\theta_0, \tag{2.1}$$

$$u = \frac{1}{P}\frac{\partial^2\theta_0}{\partial\xi^2}. \tag{2.2}$$

On the one hand, since u and h do not depend on ζ, it follows from (2.1) that $G\zeta - \partial q^*/\partial\zeta = D =$ const and

$$q^* = G\zeta^2/2 - D\zeta + D_1(\xi).$$

The second equation of (1.12) immediately shows that $D_1 =$ const

$$q^*(\zeta) - q^*(0) = G\zeta^2/2 - D\zeta. \tag{2.3}$$

On the other hand, on eliminating the unknowns h and θ_0, from Eqs. (2.1), (2.2), and (1.13) we obtain the equation

$$u^{\mathrm{IV}} - M^2 u'' + GPu = 0, \tag{2.4}$$

whose general solution is

$$u = C_1\cosh m\xi + C_2\sinh m\xi + C_3\cosh n\xi + C_4\sinh n\xi, \tag{2.5}$$

where m, n are in the general case complex parameters:

$$m = [M^2/2 + \sqrt{M^4/4 - GP}]^{1/2},$$
$$n = [M^2/2 - \sqrt{M^4/4 - GP}]^{1/2}. \tag{2.6}$$

Now with the aid of Eqs. (1.13), (2.1), and (2.3) we obtain in turn

$$h = -P_m\Big(C_1\frac{\sinh m\xi}{m} + C_2\frac{\cosh m\xi}{m} + C_3\frac{\sinh n\xi}{n} + C_4\frac{\cosh n\xi}{n} - C_5\xi - C_6\Big); \tag{2.7}$$

$$\theta_0 = \frac{1}{G}\big(C_1 n^2\cosh m\xi + C_2 n^2\sinh m\xi + C_3 m^2\cosh n\xi + C_4 m^2\sinh n\xi - M^2 C_5 - D\big). \tag{2.8}$$

For the derivation of the last formula we made use of the obvious equality $m^2 + n^2 = M^2$.

On satisfying conditions (1.15) we obtain seven equations for the determination of the constants C_i $(i = 1, 2, \ldots, 6)$ and D.

Let us dwell on certain pecularities of this system, which are connected with the boundary conditions of the initial problem.

a) In the case of free convection with the two plates unequal in temperature, i.e., for $Q^* = 0$, $\theta_{w_1} + \theta_{w_2} \neq 0$, the system has a unique solution for all values of M and G.

b) In the case of free convection with the temperature of the plates the same, i.e., for $Q^* = 0$, $\theta_{w_1} + \theta_{w_2} = 0$, the system has a solution only when a definite relation exists between G and M. The smallest possible value of G characterizes the threshold of convection.

c) In the case of mixed flow, i.e., for $Q^* \neq 0$, the system has a unique solution for arbitrary G and M.

We shall now discuss each one of these problems separately.

3. In the case that $Q^* = 0$ and $\theta_{w_1} + \theta_{w_2} \neq 0$, we first find that $C_1 = C_3 = C_5 = D = 0$, so that the temperature distribution is expressed by means of an odd function. Therefore for the case of free

stationary convection we should take $\theta_{W_1} = \theta_{W_2} = \theta_W$. Then on finding the constants C_2, C_4, and C_6 we obtain the following formulas for the velocity, for the induced field component, and for the temperature:

$$u = \frac{G\theta_w}{m^2 - n^2}\left(\frac{\sinh m\xi}{\sinh m} - \frac{\sinh n\xi}{\sinh n}\right), \quad (3.1)$$

$$h = \frac{P_m G\theta_w}{m^2 - n^2}\left(\frac{\cosh m - \cosh m\xi}{m \sinh m} - \frac{\cosh n - \cosh n\xi}{n \sinh n}\right), \quad (3.2)$$

$$\theta = \zeta + \theta_0(\xi) = \zeta + \frac{\theta_w}{m^2 - n^2}\left(\frac{n^2 \sinh m\xi}{\sinh m} - \frac{m^2 \sinh n\xi}{\sinh n}\right). \quad (3.3)$$

From these formulas it is possible to obtain the well-known result of reference 1 by means of a limiting transition for $A \to 0$, when $m \to M$ and $n \to 0$, taking into account the fact that $G\theta_W = \beta g T_W \delta^3/\nu^2$. Moreover, by noting that in the case where the ratio $M^4/4|G|P$ increases without limit we have $m/M \to 1$ and $n \to 0$, it can be easily shown that in this case the values of $u(\xi)$, $h(\xi)$, and $\theta_0(\xi)$ approach asymptotically the corresponding expressions found by Gershuni and Zhukhovitskiĭ. Therefore when $M_4/4|G|P \gg 1$ their conclusions with respect to the existence of a boundary layer, the behavior of the thermal flux, and of the induced field are approximately valid.

If the ratio $M^4/4GP$ is close to unity, then by means of a limiting transition with $m \to n$ it is possible to obtain from (3.1) — (3.3) the following approximate formulas:

$$u \approx \frac{G\theta_w}{2n \sinh^2 n}(\xi \cosh n\xi \sinh n - \sinh n\xi \cosh n), \quad (3.4)$$

$$h \approx \frac{P_m G\theta_w}{2n^3 \sinh^2 n}[(\sinh n - \xi \sinh n\xi)\, n \sinh n - (\cosh n - \cosh n\xi)(\sinh n - n \cosh n)], \quad (3.5)$$

$$\theta \approx \zeta + \theta_w\left(\frac{n}{2}\,\frac{\xi \cosh n\xi \sinh n - \sinh n\xi \cosh n}{\sinh^2 n} - \frac{\sinh n\xi}{\sinh n}\right). \quad (3.6)$$

On investigating Eq. (3.4) we find that the extremum points of the velocity profile satisfy the equation

$$n\xi \tanh n\xi = n \coth n - 1, \quad (3.7)$$

whose solution tends to $\xi = \pm 1$ when $n \to \infty$. This proves the formation of a boundary layer also in the case of comparable values of M^4 and GP.

4. When the temperature of the two plates is the same and $\theta_W = 0$, free convective motion is possible only when a certain relation holds between M and G. This relation consists of the vanishing of the determinant of the system of equations with respect to C_i and D. By bringing it to diagonal form we obtain

$$(m^2 - n^2)\sinh m \sinh n\left(\cosh m \frac{\sinh m}{n} - \cosh n \frac{\sinh n}{m}\right) = 0. \quad (4.1)$$

The only family of solutions of this equation which leads to a nontrivial solution of the problem is $n = ik\pi$ $(k = 1, 2 \ldots)$, where G must be negative.

Thus, if $G > 0$, when the temperature increases in the direction of positive z, the equilibrium of the liquid at rest is always stable. In the opposite case, when $G < 0$, the equilibrium is stable only for $|n^2| < \pi^2$, i.e., for $|GP| < \pi^2(M^2 + \pi^2)$. In the case $GP = -\pi^2(M^2 + \pi^2)$, stationary laminar flow is possible, characterized by the formulas

$$u = C\pi \sin \pi\xi, \quad h = CP_m(\cos \pi\xi + 1),$$
$$\theta = \zeta - \frac{PC}{\pi} \sin \pi\xi, \quad (4.2)$$

where C is an arbitrary constant.

The formula obtained above for the dependence of the critical value of the number G on M confirms the fact that the presence of a magnetic field significantly postpones the beginning of instability of the equilibrium. This result naturally applies only to the case of plane motion.

5. Finally, we consider the case of mixed flow when the quantity $\theta_{W_1} + \theta_{W_2}$ is arbitrary, while $Q^* \neq 0$.

In accordance with this, the solution of our problem, based on Eqs. (2.5), (2.7), and (2.8), assumes the following form

$$u = \frac{mnQ^* (\cosh m \cosh n\xi - \cosh n \cosh m\xi)}{2(m \cosh m \sinh n - n \cosh n \sinh m)} + \frac{G(\theta_{w1} + \theta_{w2})(\sinh n \sinh m\xi - \sinh m \sinh n\xi)}{2(m^2 - n^2)\sinh m \sinh n}, \quad (5.1)$$

$$h = -P_m\left[\frac{Q^*(m \cosh m \sinh n\xi - n \cosh n \sinh m\xi)}{2(m \cosh m \sinh n - n \cosh n \sinh m)} + \frac{G(\theta_{w1} + \theta_{w2})(n \sinh n \cosh m\xi - m \sinh m \cosh n\xi)}{2mn(m^2 - n^2)\sinh m \sinh n} - \frac{Q^*\xi}{2} - \frac{G(\theta_{w1} + \theta_{w2})}{2(m^2 - n^2)mn}(n \coth m - m \coth n)\right], \quad (5.2)$$

$$\theta = \zeta + \frac{1}{G}\left[\frac{mnQ^*(m^2 \cosh m \cosh n\xi - n^2 \cosh n \cosh m\xi)}{2(m \cosh m \sinh n - n \cosh n \sinh m)} + \frac{G(\theta_{w1} + \theta_{w2})(n^2 \sinh n \sinh m\xi - m^2 \sinh m \sinh n\xi)}{2(m^2 - n^2)\sinh m \sinh n} - \frac{mn(m^2 - n^2)Q^*}{2(m \tanh n - n \tanh m)} + \frac{G(\theta_{w2} - \theta_{w1})}{2}\right]. \quad (5.3)$$

When $G \rightarrow 0$, $\theta_{wi} \rightarrow 0$, the formulas (5.1) and (5.2) reduce to the well-known solution of Hartmann's problem.[3]

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33