# ON THE LAGRANGIAN FORMALISM FOR SPIN VARIABLES

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Submitted to JETP editor January 29, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 170-178 (July, 1959)

It is shown that a change of the class of allowable variations in the Schwinger variation principle[1] makes it possible to include the spin variables in the general Lagrangian formalism in both the nonrelativistic and relativistic cases.

THE transition from classical mechanics to quantum mechanics is usually made by means of the correspondence principle, which consists of replacement of the classical Poisson brackets by the quantum commutation relations. Despite the great generality of the correspondence principle, it has well-known limitations, owing to the requirement that a classical Lagrangian function exist for the system in question. Because of this limitation the spin variables, which, as is well known, have no classical analogues, are introduced into the general scheme of quantum mechanics not on the basis of the correspondence principle, but by means of a formal increase of the number of components of the wave function and the requirement that under transformations of rotation these components shall transform according to some irreducible representation of the rotation group.

Schwinger has proposed a Lagrangian formulation of quantum mechanics, based on the application of a variation principle to an operator action function.[1] Because of the operator character of the Schwinger variation principle, the existence of a classical (nonoperator) Lagrangian function is not a necessary condition for its use. In this sense the Schwinger variation principle is broader than the correspondence principle.

Owing, however, to the limitations imposed by Schwinger on the class of allowable variations,* the spin variables cannot be included in this more general scheme.

In the present paper it is shown that the limitations on the class of allowable variations postulated by Schwinger are not necessary (cf. also reference 2). A change of the class of allowable variations makes it possible to include the spin variables (for an arbitrary value of the spin) in the general scheme of the Schwinger variation principle, both in the nonrelativistic case and also in the relativistic case. The Lagrangian formalism for the spin variables in the relativistic case leads to a natural introduction of the proper time into the theory.

*Schwinger restricted the class of allowable variations to those commuting or anticommuting with all other operators. Such a limitation seemed to Schwinger to be necessary for the obtaining of the equations of motion from the condition that the action integral be stationary.

## 1. THE NONRELATIVISTIC CASE

In the nonrelativistic case we shall describe the spin variables by a vector $\mathbf{s}$. We shall determine the equations of motion and the operator properties of the vector $\mathbf{s}$ on the basis of the application of the operator action principle to the following Lagrangian function:

$$L = \tfrac{1}{2} i [\mathbf{s}, \dot{\mathbf{s}}] - H(\mathbf{s}), \tag{1}$$

where $H(\mathbf{s})$ is an arbitrary function of $\mathbf{s}$, which is the Hamiltonian of the system, $\dot{\mathbf{s}}$ is the time derivative of the operator $\mathbf{s}$ in the Heisenberg representation, and the square brackets denote the commutator (we shall use a system of units in which $\hbar = 1$).

The first term in the expression (1) corresponds to the kinematic part of the Lagrangian function. This term is uniquely determined from the condition of invariance under rotation transformations and the requirement that the equations of motion for the spin vector be equations of the first order.

We note that in the classical case, in which the components of the vector $\mathbf{s}$ are c-numbers and not operators, the scalar product $\mathbf{s}\dot{\mathbf{s}}$ is a total derivative with respect to time, so that the addition of such a term to the Lagrangian function does not enable us to get equations of motion for the spin vector in the classical case.

An arbitrary variation of the action integral for this Lagrangian function can be written in the form

$$\delta W_{21} = \delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \{i[\delta^* s, \dot{s}] - \delta^* H\} dt + \left[\tfrac{1}{2} i [s, \delta^* s] + L\delta t\right]_{t_1}^{t_2}. \quad (2)$$

According to the Schwinger variation principle[1] the variation $\delta W_{21}$ for fixed "external" parameters has the form

$$\delta W_{21} = G(t_2) - G(t_1), \quad (3)$$

where the operators $G(t_1)$ and $G(t_2)$ depend only on the dynamical variables referring to the times $t_1$ and $t_2$, respectively, and are interpreted as operators generating infinitesimal unitary transformations corresponding to the variations in question; that is, for a fixed time:*

$$\delta^* s = i[G, s], \quad (4)$$

where

$$G = \tfrac{1}{2} i [s, \delta^* s] + L\delta t. \quad (5)$$

The last expression, which defines the explicit form of the generating operator $G(t)$, is obtained by comparing Eqs. (2) and (3) under the condition that

$$\int_{t_1}^{t_2} \{i[\delta^* s, \dot{s}] - \delta^* H\} dt = 0. \quad (6)$$

The variation $\delta^* \mathbf{s}$ in the formulas (2), (4), (5), and (6) does not contain the changes caused by the variation $\delta t$. The total variation, including these changes also, has the form:

$$\delta s = \delta^* s + \dot{s}\delta t. \quad (7)$$

(We note that in the interior of the region of integration [Eq. (2)] the time $t$ is not varied, so that the variation $\delta \mathbf{s}$ coincides with the variation $\delta^* \mathbf{s}$.) Using Eq. (7), we rewrite Eq. (5) in the form

$$G = G_s + G_t,$$

where

$$G_s = \tfrac{1}{2} i [s, \delta s], \qquad G_t = -H\delta t \quad (5')$$

and

$$\delta s - \dot{s}\delta t = i[G_s + G_t, s]. \quad (4')$$

*The limitation to variations that commute or anticommute has the consequence that relations of the type (4) are not always satisfied for the general case of quantum-mechanical variables. The difficulties that thus arise in the interpretation of the derivatives of operators have been noted in a paper by Burton and Touschek.[3] The removal of these difficulties required the bringing in of additional nontrivial arguments.[4] The broadening of the class of allowable variations makes it possible to obtain a unique interpretation of the derivatives of operators, without the difficulties that have been mentioned.

Because of the independence of the variations $\delta \mathbf{s}$ and $\delta t$, the relation (4′) separates into two independent relations

$$\delta s = i[G_s, s], \quad (8)$$

$$\dot{s} = i[H, s]. \quad (9)$$

The relation (9) defines the equation of motion for the vector $\mathbf{s}$.

To establish the commutation relations between the different components of the vector $\mathbf{s}$, let us consider the rotation transformation

$$s_i \to s_i + \varepsilon_{ikl}\omega_k s_l \ .$$

($\epsilon_{ikl}$ is the antisymmetric unit tensor).

We obtain the generating operator $G_\omega$ for this transformation by substituting

$$\delta s_i' = \varepsilon_{ikl}\omega_k s_l$$

in the general expression (5′):

$$G_\omega = -\tfrac{1}{2} i\omega_i \varepsilon_{ikl} [s_k, s_l]. \quad (10)$$

In virtue of (8), the commutator of the operator $G_\omega$ with an arbitrary vector function $\mathbf{A}(\mathbf{s})$ (under the condition that no "external" vectors are involved in this function) is given by

$$[G_\omega, A_i(s)] = -i\varepsilon_{ikl}\omega_k A_l(s),$$

or, since the vector $\boldsymbol{\omega}$ is arbitrary, we have

$$[M_i, A_k(s)] = i\varepsilon_{ikl} A_l(s), \quad (11)$$

where

$$M_i = -\tfrac{1}{2} i\varepsilon_{ikl}[s_k, s_l]. \quad (12)$$

If we take as the vector $\mathbf{A}$ the vector $\mathbf{M}$ defined by (12), then Eq. (11) leads to the well-known commutation relations for the components of the angular momentum operator,

$$[M_i, M_k] = i\varepsilon_{ikl} M_l. \quad (13)$$

This also establishes the well-known connection between the operator for an infinitesimal rotation and the angular momentum operator,

$$G_\omega = \boldsymbol{\omega}\mathbf{M}. \quad (14)$$

The vector $\mathbf{s}$ is connected with the angular momentum vector by the relation (12). Using Eqs. (12) and (13) together with the relation

$$[M_i, s_k] = i\varepsilon_{ikl} s_l, \quad (15)$$

which is a special case of Eq. (11) for $\mathbf{A} = \mathbf{s}$, and assuming that the operator $\mathbf{s}$ acts in a space with a finite number of dimensions, one can easily show that

$$s = M, \quad (16)$$

i.e., the operator properties of the vector **s** coincide with those of the angular momentum operator

$$[s_i, s_k] = i\varepsilon_{ikl}s_l. \quad (17)$$

Thus the vector **s** actually describes the spin degrees of freedom of the particle.

In the derivation of the equations of motion (9) and the commutation relations (17), we have used only the properties of the generating operator G [Eqs. (4) and (8)]. We have established the explicit form of the operator G on the basis of the assumption that the variation of the action integral vanishes in the entire internal region of integration in the interval between $t_1$ and $t_2$ [Eq. (3)].

A little later we shall show that this assumption is satisfied as a consequence of the equations of motion (9) if the variation $\delta\mathbf{s}$ we are considering belongs to a definite class.

For the definition of the class of allowable variations, let us adopt the natural requirement that the operator

$$G_{\mathbf{s}}(t) = \tfrac{1}{2} i [\mathbf{s}(t), \delta\mathbf{s}(t)] \quad (18)$$

is to be the generating operator for the change $\delta\mathbf{s}(t)$; i.e., that

$$\delta\mathbf{s}(t) = i [G_{\mathbf{s}}(t), \mathbf{s}(t)] \quad (19)$$

also for variations in the entire internal region of integration.

Substituting Eq. (19) in Eq. (18), we get the relation

$$G_{\mathbf{s}} = -\tfrac{1}{2}[\mathbf{s}, [G_{\mathbf{s}}, \mathbf{s}]], \quad (20)$$

in which, for brevity, we have omitted the variable t. To find the general form of the generating operator $G_{\mathbf{s}}$ compatible with Eq. (20), let us expand $G_{\mathbf{s}}$ in terms of the irreducible tensors $Y^{lm}(\mathbf{s})$

$$G_{\mathbf{s}} = \sum_{l,m} a_{lm} Y^{lm}(\mathbf{s}).$$

Using the commutation rules for the vector **s**, one can easily show that

$$[\mathbf{s}, [\mathbf{s}, Y^{lm}(\mathbf{s})]] = l(l+1) Y^{lm}(\mathbf{s}). \quad (21)$$

The relation (21) is compatible with Eq. (20) only for $l = 1$, from which it follows that the most general expression for the operator $G_{\mathbf{s}}$ has the form

$$G_{\mathbf{s}} = \sum_{m=-1}^{1} a_{1m} Y^{1m}(\mathbf{s}) = \boldsymbol{\omega}\mathbf{s}. \quad (22)$$

Substituting Eq. (22) in Eq. (19), we get the general expression characterizing the class of allowable variations,

$$\delta s_i(t) = i[\omega_k(t) s_k(t), s_i(t)] = \varepsilon_{ikl}\omega_k(t) s_l(t), \quad (23)$$

where the $\omega_k(t)$ are arbitrary functions of the time.

We note that the variations $\delta\mathbf{s}$ that we used for the derivation of the commutation relations also belong to this class of variations.

We shall now show that the principle of stationary action, which is the condition that the integral in Eq. (2) vanish,

$$\int_{t_1}^{t_2} \{i[\delta\mathbf{s}, \dot{\mathbf{s}}] - \delta H\}\, dt = 0, \quad (24)$$

is satisfied for the class of allowable variations that we are considering.

In fact, if the variation $\delta H$ occurs only on account of variation of the vector **s**, it can be put in the form

$$\delta H = i[G_{\mathbf{s}}, H] = -\tfrac{1}{2}[[\mathbf{s}, \delta\mathbf{s}], H]. \quad (25)$$

Using the operator Jacobi identity, we can write the last expression in another form

$$\delta H = -\tfrac{1}{2}[\mathbf{s}, [\delta\mathbf{s}, H]] - \tfrac{1}{2}[\delta\mathbf{s}, [H, \mathbf{s}]]. \quad (26)$$

For an arbitrary operator X we have as a consequence of Eq. (23) the relation

$$s_i X \delta s_i = s_i X \varepsilon_{ikl}\omega_k s_l = -\delta s_i X s_i, \quad (27)$$

which enables us to transform the first term in the expression (26) to the form

$$-\tfrac{1}{2}[\mathbf{s}, [\delta\mathbf{s}, H]] = \tfrac{1}{2}[\delta\mathbf{s}, [\mathbf{s}, H]]. \quad (28)$$

Substituting Eq. (28) in Eq. (26), we get the following expression for $\delta H$

$$\delta H = -[\delta\mathbf{s}, [H, \mathbf{s}]], \quad (29)$$

and a corresponding expression for the integral in Eq. (24),

$$\int_{t_1}^{t_2} i[\delta\mathbf{s}, \dot{\mathbf{s}} - i[H, \mathbf{s}]]\, dt. \quad (30)$$

It can be seen at once from Eq. (30) that the principle of stationary action for the class of allowable variations under consideration is a consequence of the equations of motion (9).

The results of this section show that the information about the equations of motion and the operator properties of the vector **s** obtained by the application of the operator action principle to the Lagrangian function (1) are internally consistent and correctly describe the properties of the spin variables.

## 2. THE RELATIVISTIC CASE

The Schwinger variation principle permits us to introduce the spin variables into the Lagrangian formalism also in the relativistic case.

As the natural relativistic generalization of the spin vector **s** let us consider the four-dimensional vector $\Gamma_\mu$. We denote the coordinate and momentum four-vectors by $x_\mu$ and $p_\mu$, respectively. Relativistic invariance of the formulation of the theory can be achieved by the introduction of an invariant parameter $\tau$, on which the dynamical variables $\Gamma_\mu$, $x_\mu$, $p_\mu$ depend, and which we shall call hereafter the "proper" time of the particle.*

The operator Lagrangian function, on the basis of which one obtains the equations of motion and the commutation relations for the dynamical variables $\Gamma_\mu$, $x_\mu$, $p_\mu$, is now chosen in the form†

$$\mathcal{L} = \tfrac{1}{8} i[\Gamma_\mu, \dot{\Gamma}_\mu] - \tfrac{1}{4}(\{x_\mu, \dot{p}_\mu\} - \{\dot{x}_\mu, p_\mu\}) - \mathcal{H}, \tag{31}$$

where $\mathcal{H}$ is a certain function of $\Gamma_\mu$, $x_\mu$, $p_\mu$; the dot denotes differentiation with respect to the proper time $\tau$, and the braces denote anticommutators. The kinematic part of the Lagrangian function is written in relativistically invariant form.

The action operator corresponding to the Lagrangian function (31) is given by

$$W_{21} = \int_{\tau_1}^{\tau_2} \mathcal{L}(\tau)\, d\tau. \tag{32}$$

Varying both the dynamical variables and the proper-time parameter in the action integral, we get

$$\delta W_{21} = \int_{\tau_1}^{\tau_2} d\tau \left(\tfrac{1}{4} i [\delta\Gamma_\mu, \dot{\Gamma}_\mu] - \tfrac{1}{2}\{\delta x_\mu, \dot{p}_\mu\} + \tfrac{1}{2}\{\delta p_\mu, \dot{x}_\mu\} - \delta\mathcal{H}\right) + G(\tau_2) - G(\tau_1), \tag{33}$$

where

$$G(\tau) = G_\Gamma + G_{xp} + G_\tau, \quad G_\Gamma = \tfrac{1}{8} i [\Gamma_\mu, \delta\Gamma_\mu],$$
$$G_{xp} = \tfrac{1}{4}(\{\delta x_\mu, p_\mu\} - \{\delta p_\mu, x_\mu\}), \quad G_\tau = -\mathcal{H}\delta\tau. \tag{34}$$

Just as in the preceding section, the equations of motion and the commutation relations for the quantities $\Gamma_\mu$, $x_\mu$, $p_\mu$ are obtained by means of the operators G that generate infinitesimal unitary transformations of the dynamic variables:

$$i[G, \Gamma_\mu] = \delta\Gamma_\mu - \dot{\Gamma}_\mu \delta\tau, \qquad i[G, x_\mu] = \delta x_\mu - \dot{x}_\mu\delta\tau,$$
$$i[G, p_\mu] = \delta p_\mu - \dot{p}_\mu \delta\tau. \tag{35}$$

In Eqs. (34) and (35) $\delta\Gamma_\mu$, $\delta x_\mu$, $\delta p_\mu$ are the total variations of the quantities $\Gamma_\mu$, $x_\mu$, $p_\mu$.

The allowable class of variations is defined by the requirement that (35) be compatible with the principle of stationary action, i.e., with the assumption that for arbitrary $\mathcal{H}$ the variation $\delta W_{21}$ of Eq. (33) have the form‡

$$\delta W_{21} = G(\tau_2) - G(\tau_1). \tag{33'}$$

For the Lagrangian function (31) this class of variations has the form

$$\delta\Gamma_\mu = \varepsilon_{\mu\nu}\Gamma_\nu, \qquad \delta x_\mu = A_{\mu\nu}x_\nu + B_{\mu\nu}p_\nu,$$
$$\delta p_\mu = -A_{\nu\mu}p_\nu + C_{\mu\nu}x_\nu, \tag{36}$$

where

$$\varepsilon_{\mu\nu} = \varepsilon^*_{\mu\nu} = -\varepsilon_{\nu\mu}, \qquad B_{\mu\nu} = B^*_{\mu\nu} = B_{\nu\mu}, \qquad C_{\mu\nu} = C^*_{\mu\nu} = C_{\nu\mu},$$
$$A_{\mu\nu} = A^*_{\mu\nu}, \tag{37}$$

for $\mu, \nu = 0, 1, 2, 3$. The matrices $\epsilon$, A, B, C have arbitrary time dependences.

Since the variations $\delta\tau$ and $\delta\Gamma_\mu$, $\delta x_\mu$, $\delta p_\mu$ are independent, Eqs. (34) and (35) lead to the following equations of motion for the quantities $\Gamma_\mu$, $x_\mu$, $p_\mu$:

$$\dot{\Gamma}_\mu = i^{-1}[\Gamma_\mu, \mathcal{H}], \qquad \dot{p}_\mu = i^{-1}[p_\mu, \mathcal{H}],$$
$$\dot{x}_\mu = i^{-1}[x_\mu, \mathcal{H}]. \tag{38}$$

It can be seen from these equations that $\mathcal{H}$ plays the part of the Hamiltonian with respect to the proper time (and not with respect to the ordinary time, as in the nonrelativistic case).

Let us now verify that with the class of allowable variations (36) the principle of stationary action is satisfied in virtue of the equations of motion (38).

Using Eqs. (36) and (37), by means of the generating operator $G_\Gamma + G_{xp}$ we can transform the variation $\delta\mathcal{H}$ to the form:

$$\delta\mathcal{H} = \delta_\Gamma\mathcal{H} + \delta_{xp}\mathcal{H},$$

where

$$\delta_\Gamma\mathcal{H} = i[G_\Gamma, \mathcal{H}] = \tfrac{1}{4}[\delta\Gamma_\nu, [\Gamma_\nu, \mathcal{H}]], \quad \delta_{xp}\mathcal{H} = i[G_{xp}, \mathcal{H}]$$
$$= \tfrac{1}{2} i^{-1}\{\delta x_\mu, [\mathcal{H}, p_\mu]\} - \tfrac{1}{2} i^{-1}\{\delta p_\mu, [\mathcal{H}, x_\mu]\}. \tag{39}$$

Substituting Eq. (39) in Eq. (33), we easily verify that the principle of stationary action in the form (33′) is in fact satisfied.

Let us go on to the determination of the commutation relations for the operators of the dynamical variables.

By Eq. (36) the variation $\delta\Gamma_\mu$ is independent of the variations $\delta x_\mu$, $\delta p_\mu$, so that the first of

*Wave equations with the proper time have been considered by Fock, Nambu, Feynman, and Schwinger,[5-9] but the Lagrangian formulation for the equations of motion with the proper time was carried through only for the classical case (cf., e.g., reference 5) and for the Klein-Gordon equation.[7]

†In the present section we use a system of units in which $\hbar = c = 1$.

‡A detailed investigation of the class of allowable variations will be made in another paper. Here we note only that the class of allowable variations preserves the invariance of the kinematic part of the Lagrangian function.

the formulas (35) can be written in the form

$$\delta\Gamma_\mu = i[G_\Gamma, \Gamma_\mu] = \tfrac{1}{8}[\Gamma_\mu, [\Gamma_\nu, \delta\Gamma_\nu]].$$

Using the formulas (36) and (37), which define the allowable variations $\delta\Gamma_\mu$, we easily obtain the following commutation relations for the quantities $\Gamma_\mu$:*

$$\tfrac{1}{4}[\Gamma_\nu, [\Gamma_\mu, \Gamma_\lambda]] = \delta_{\mu\nu}\Gamma_\lambda - \delta_{\lambda\nu}\Gamma_\mu. \quad (40)$$

If we introduce the notation

$$I_{\mu\lambda} = \tfrac{1}{4}[\Gamma_\mu, \Gamma_\lambda], \quad (41)$$

Equation (40) takes the form

$$[\Gamma_\nu, I_{\mu\lambda}] = \delta_{\mu\nu}\Gamma_\lambda - \delta_{\lambda\nu}\Gamma_\mu. \quad (42)$$

If follows at once from Eqs. (41) and (42) that

$$[I_{\mu\lambda}, I_{\sigma\nu}] = \delta_{\mu\sigma}I_{\nu\lambda} + \delta_{\lambda\sigma}I_{\mu\nu} + \delta_{\lambda\nu}I_{\sigma\mu} + \delta_{\mu\nu}I_{\lambda\sigma}. \quad (43)$$

Thus the $I_{\mu\lambda}$ are the infinitesimal operators of the four-dimensional rotation group, and according to Eqs. (34) and (36) the generating operator for four-dimensional rotations has the form

$$G_\Gamma \equiv G_\varepsilon = \tfrac{1}{2} i\varepsilon_{\mu\nu}I_{\mu\nu}. \quad (44)$$

The commutation relations between the $x_\mu$ and $p_\mu$ are obtained by means of the generating operator $G_{xp}$ on the class of allowable variations (36) in the same way as the commutation relations for the $\Gamma_\mu$, and have the following rather complicated form

$$2\delta_{\lambda\mu}x_\nu = i^{-1}[x_\mu, \{x_\nu, p_\lambda\}], \qquad 2\delta_{\lambda\mu}p_\nu = -i^{-1}[p_\mu, \{x_\lambda, p_\nu\}],$$
$$[p_\mu, \{p_\nu, p_\lambda\}] = 0, \quad \delta_{\lambda\mu}p_\nu + \delta_{\nu\mu}p_\lambda = \tfrac{1}{2}i^{-1}[x_\mu, \{p_\nu, p_\lambda\}],$$
$$[x_\mu, \{x_\nu, x_\lambda\}] = 0, \quad \delta_{\lambda\mu}x_\nu + \delta_{\nu\mu}x_\lambda = -\tfrac{1}{2}i^{-1}[p_\mu, \{x_\nu, x_\lambda\}]. \quad (45)$$

These commutation relations are compatible with each other and with the usual commutation relations between $x_\mu$ and $p_\nu$,

$$[p_\mu, p_\nu] = 0, \quad [x_\mu, x_\nu] = 0, \quad [x_\mu, p_\nu] = i\delta_{\mu\nu}. \quad (46)$$

These latter relations can be obtained directly by means of the generating operators if we add to the Lagrangian function (31) the total derivative $\frac{1}{4}(d/d\tau)\{x_\mu, p_\mu\}$.[4]

The kinematic term of the altered Lagrangian function, depending on $x_\mu$, $p_\nu$, then takes the form

$$\tfrac{1}{2}\{\dot{x}_\mu, p_\mu\}. \quad (47)$$

It follows from Eq. (47) that the new generating operator $G'_{xp}$ is defined by the equation

$$G'_{xp} = \tfrac{1}{2}\{\delta x_\mu, p_\mu\}. \quad (48)$$

The class of allowable variations for the altered Lagrangian function has the form†

$$\delta\Gamma_\mu = \varepsilon_{\mu\nu}\Gamma_\nu, \quad \delta x_\mu = \varepsilon_\mu, \quad \delta p_\mu = 0. \quad (49)$$

With this class of variations the principle of stationary action in the form (33′) is also satisfied. From Eqs. (48) and (49) and the independence of the variations $\delta\Gamma_\mu$ and $\delta x_\mu$, we get the following commutation relations:

$$[p_\nu, p_\mu] = 0, \quad [x_\mu, p_\nu] = i\delta_{\mu\nu}, \quad [\Gamma_\mu, p_\nu] = 0. \quad (50)$$

If we add to the Lagrangian function (31) the total derivative $-\frac{1}{4}(d/d\tau)\{x_\mu, p_\mu\}$, then in a similar way we get two new commutation relations:

$$[x_\mu, x_\nu] = 0, \quad [\Gamma_\mu, x_\nu] = 0. \quad (51)$$

Equations (42), (43), (50), and (51) are the final form of the commutation relations between $\Gamma_\mu$, $x_\mu$, $p_\mu$.

All of the operators considered in the present section are taken in the Heisenberg representation, in which the state vector $\psi$ does not depend on the proper time $\tau$.

In the Schrödinger representation, in which the operators do not depend on the proper time $\tau$, the state vector satisfies the following Schrödinger equation:

$$i\partial\psi(\tau)/\partial\tau = \mathcal{H}\psi(\tau). \quad (52)$$

For a free particle the Lagrangian function must be invariant with respect to displacements of the four-vector $x_\mu$. The only relativistically invariant combination that satisfies this requirement and does not contain any dimensional constants is the quantity $\Gamma_\mu p_\mu$. Therefore it is natural to take as the function $\mathcal{H}$ for the case of free particles:

$$\mathcal{H} = -\tfrac{1}{2}\{\Gamma_\mu, p_\mu\}. \quad (53)$$

Then in virtue of the commutation relations the equations of motion (8) of the Heisenberg representation take the form

$$\dot{\Gamma}_\mu = 4iI_{\mu\nu}p_\nu, \quad \dot{x}_\mu = -\Gamma_\mu, \quad \dot{p}_\mu = 0. \quad (54)$$

The equations of motion and the commutation

*The algebra of the matrices $\Gamma_\mu$, as defined by the commutation relations (40), has been studied in detail by Bhabha.[10] If we include in the kinematic part of the Lagrangian function arbitrary functions of the operators $\Gamma_\mu$ of the forms $f_\mu(\Gamma)\dot{\Gamma}_\mu$, then Eqs. (42) and (43) remain unchanged, and Eq. (41) takes a more general form, which depends on the concrete form of the functions $f_\mu(\Gamma)$. This corresponds to the passage to an algebra that arises in the consideration of general relativistically invariant equations (cf., e.g., reference 11).

†These variations also leave invariant the kinematic part of the altered Lagrangian function (cf. footnote ‡, p. 124).

relations in the presence of an electromagnetic field are obtained from the Lagrangian function (31) with the Hamiltonian

$$\mathscr{H} = -\tfrac{1}{2}\{\Gamma_\mu, \pi_\mu\}, \qquad \pi_\mu = p_\mu - eA_\mu$$

and are

$$\dot{\Gamma}_\mu = 4iI_{\mu\nu}\pi_\nu, \quad \dot{x}_\mu = -\Gamma_\mu, \quad \dot{\pi}_\mu = -eF_{\mu\nu}\Gamma_\nu,$$
$$[\pi_\mu, \pi_\nu] = ieF_{\mu\nu}, \quad [\pi_\mu, x_\nu] = -i\delta_{\mu\nu}, \quad [x_\mu, x_\nu] = 0. \tag{55}$$

The commutation relations for the matrices $\Gamma_\mu$ are those shown in Eq. (40). The Hamiltonian $\mathcal{H}$ does not depend explicitly on $\tau$, so that there exist solutions of Eq. (52) in the form[7]

$$\psi(\tau) = e^{-im\tau}\psi,$$

where $\psi$ satisfies the equation

$$\mathscr{H}\psi = m\psi. \tag{56}$$

For the case of free particles, Eq. (56) can be written in the form

$$(\Gamma p + m)\psi = 0 \tag{57}$$

and coincides with the relativistically invariant equations studied by Bhabha[10] (cf. footnote* p. 125).

In conclusion, the writers express their gratitude to A. I. Akhiezer and P. I. Fomin for helpful discussions.

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Translated by W. H. Furry
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