SOME APPLICATIONS IN THE THEORY OF METALS OF THE METHOD OF SUMMATION OF THE MAIN FEYNMAN DIAGRAMS

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Starting from the condition that the average phonon energy must be small compared to the average energy of the electron transitions, we have constructed a Hamiltonian with a direct electron-electron interaction which describes in the given approximation a Fröhlich system of interacting electrons and phonons.

THE Fröhlich model has recently received widespread interest; this model is described by the Hamiltonian

$$H = H_{el} + H_{ph} + H_{int};$$
$$H_{el} = \sum_{ks} \varepsilon(k)\, a_{k,s}^{+} a_{k,s}; \qquad H_{ph} = \sum_{q} \omega(q)\, b_q^{+} b_q;$$
$$H_{int} = \sum_{kqs} g(q) \sqrt{\frac{\omega(q)}{2V}}\, a_{k,s}^{+} a_{k+q,s} (b_q^{+} + b_{-q}). \tag{1}$$

A number of important physical results can, however, be obtained from a simpler model without phonons, the Hamiltonian of which is of the form:[1,2]

$$H_1 = H_{el} + H'_{int};$$
$$H'_{int} = -\frac{1}{2V} \sum_{kk'qss'} I(kk'q)\, a_{k,s}^{+} a_{k+q,s} a_{k',s'}^{+} a_{k'-q,s'}, \tag{2}$$

where $I(kk'q)$ is a function which decreases sufficiently fast when one gets away from the Fermi surface. The conditions, however under which the models described by (1) and (2) can be considered to be approximately the same are still not completely clear, notwithstanding a number of studies which have been made.[1,2] Moreover, there does not exists a sufficiently consistent way of determining the function $I(kk'q)$ in (2).

The method of summing the main Feyman diagrams makes it possible to construct the Hamiltonian (2) from the Hamiltonian (1) under only one assumption:

$$\overline{\omega} \ll \overline{\Delta\varepsilon} \tag{3}$$

(average phonon energy negligibly small compared to the average energy of the electron transitions).

Performing the usual procedure of renaming the creation and annihilation operators of the electrons inside the Fermi sphere, we go after that over to the "interaction representation" where the part of the Hamiltonian of non-interacting fields will be played by the quadratic form

$$H_0 = \sum_q \omega(q)\, b_q^{+} b_q + \sum_{ks} \tilde{\varepsilon}(k)\, a_{k,s}^{+} a_{k,s},$$

where

$$\tilde{\varepsilon}(k) = \begin{cases} \varepsilon(k) & (|k| > k_F) \\ -\varepsilon(k) & (|k| < k_F). \end{cases} \tag{4}$$

In the following we shall in accordance with reference 2 give our discussion along the lines of the covariant formulation of the quantum theory of fields. The S matrix of the system is of the form

$$S = S_{-\infty}^{0} = T\left[\exp\left(-i \int_{-\infty}^{0} H_{int}(\theta)\, d\theta\right)\right]. \tag{5}$$

The energy of the interaction of the electrons with the phonons, which does not depend on the time, is, if we take all radiative corrections into account, equal to

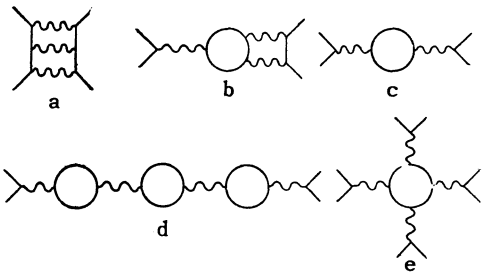
$$\hat{R} = T\left[H_{int}(0) \exp\left(-i \int_{-\infty}^{0} H_{int}(\theta)\, d\theta\right)\right]. \tag{6}$$

The chronological pairs which occur when $\hat{R}$ is written out explicitly are written in the usual way:

$$\dot{B}_q(t)\, \dot{B}_{q'}(t') = [\theta(t-t')\, e^{i\omega(q)(t'-t)} + \theta(t'-t)\, e^{i\omega(q)(t-t')}]\, \delta_{q+q',0};$$
$$B_q(t) = b_q^{+}(t) + b_{-q}(t);$$
$$\dot{a}_{k,s}(t)\, \dot{a}_{k',s'}^{+}(t') = -\dot{a}_{k',s'}^{+}(t')\, \dot{a}_{k,s}(t)$$
$$= \delta_{kk'}\delta_{ss'} \begin{cases} \theta(t-t')\, e^{-i\tilde{\varepsilon}(k)(t-t')} & (|k| > k_F) \\ -\theta(t'-t)\, e^{i\tilde{\varepsilon}(k)(t-t')} & (|k| < k_F) \end{cases} \tag{7}$$

(to attain more compact formulae we have marked the operators corresponding to a chronological pairing everywhere by the same number of dots on top).

The effective Hamiltonian with a direct electron-electron interaction is a sum of terms, each of which is described by a Feynman diagram with external electron lines (diagrams of the kind illus-



trated in the figure). Part of the diagrams corresponds to triple, quadruple, ... interactions between electrons. We shall retain in the operator series $\hat{R}$ the terms with coefficients of the lowest order in ω, i.e., such terms where the number of integrations cancelling in the denominator 2ω, is as large as possible for a given order in g^2. This leads to the following basic rule of selecting the main diagrams: in each term of the expansion of (6) in g^2 those diagrams must remain for which the number of possible ways of breaking them into two parts by lines intersecting only phonon lines is as large as possible. We shall denote the contribution of the diagrams with external electron lines by $\hat{R}^{(el)}$ and we shall in the following use (3) to look for a formal approximate expression for $\hat{R}^{(el)}$. It is clear that one can at once drop diagrams of the type a, b, in the Fig.

Using the relations

$$\omega(-q) = \omega(q), \qquad g(-q) = g(q) \tag{8}$$

all integrations over the time in terms corresponding to diagrams of the type c, d, and e can be performed simply enough explicitly and the result of each integration is the appearance of the corresponding energy denominator. This makes it possible to formulate two more rules: one must discard all diagrams with more than four external electron lines (type e), and from diagrams of the type c or d only retain those for which the pair of external electron lines starts from a vertex corresponding to the first factor under the T-product sign in (6).

The correctness of all rules mentioned is illustrated below by the example of evaluating the first three terms in the expansion of $\hat{R}^{(el)}$ in g^2. We have

$$\hat{R}_0^{(el)} = -\frac{i}{2V}\int_{-\infty}^{0} dt \sum_{kk'qq'ss'} g(q)\,g(q')\sqrt{\omega(q)\,\omega(q')}\,a^+_{k,s}(0)$$
$$\times a_{k+q,s}(0)\,\dot{B}_q(0)\,a^+_{k',s'}(t)\,a_{k'+q',s'}(t)\,\dot{B}_{q'}(t)$$
$$= -\frac{1}{2V}\sum_{kk'qss'}\frac{g^2(q)\,\omega^2(q)}{-z^2(k',q)+\omega^2(q)}$$
$$\times a^+_{k,s}(0)\,a_{k+q,s}(0)\,a^+_{k',s'}(0)\,a_{k'-q,s'}(0); \tag{9}$$

$$z(k',q) = \varepsilon(k') - \varepsilon(k'-q); \tag{10}$$

(the N-product signs are here and everywhere in the following omitted).

In evaluating the further terms we shall all the time meet with the factor

$$C(t',t) = \int_{-\infty}^{0} dt_1 \sum_{k_1k_2s_1s_2q_1q_2} \dot{B}_{q'}(t')\,\ddot{a}^+_{k_1,s_1}(t_1)$$
$$\times \dddot{a}_{k_1+q_1,s_1}(t_1)\,\dot{B}_{q_1}(t_1)\,\dddot{a}^+_{k_2,s_2}(t)\,\ddot{a}_{k_2+q_2,s_2}(t)\,B_{q_2}(t)$$
$$= 2\sum_q \sum_k{}^{(q)}\Big\{-\frac{e^{i\omega t'+iYt}}{i(\omega+Y)} + \frac{2\theta(t'-t)}{i(Y^2-\omega^2)}\left[Ye^{i\omega(t-t')} - \omega e^{iY(t-t')}\right]$$
$$+ \frac{2\theta(t-t')}{i(Y^2-\omega^2)}\left[Ye^{i\omega(t'-t)} - \omega e^{iY(t'-t)}\right]\Big\} B_{-q}(t)$$
$$= 2\sum_k{}^{(q)} C_{k,q}(t',t)\,B_{-q}(t);$$

$$\omega = \omega(q), \qquad Y = Y(k,q) = \tilde{\varepsilon}(k+q) + \tilde{\varepsilon}(k). \tag{11}$$

The symbol $\sum_k^{(q)}$ indicates a summation over k where $|k+q| > k_F$; $|k| < k_F$.

We shall now evaluate the term $\hat{R}_1^{(el)}$ which corresponds to a diagram with one electron-hole loop (type c):

$$\hat{R}_1^{(el)} = \int_{-\infty}^{0}\int_{-\infty}^{0} dt\,dt_1 \sum_{kk'qq'ss'} \frac{g^3(q)\,g(q')\sqrt{\omega^3(q)\,\omega(q')}}{(2V)^2}\,a^+_{k,s}(0)$$
$$\times a_{k+q,s}(0)\,a^+_{k',s'}(t_1)\,a_{k'+q',s'}(t_1)\,\dot{B}_{q'}(t_1)\,\dot{C}(0t)$$
$$\approx -\frac{2}{(2V)^2}\sum_{kk'qss'}\frac{g^4(q)\,\omega^2(q)}{-z^2(k',q)+\omega^2(q)}\sum_{k_1}{}^{(q)}\frac{1}{\tilde{\varepsilon}(k_1)+\tilde{\varepsilon}(k_1-q)}$$
$$\times a^+_{k,s}(0)\,a_{k+q,s}(0)\,a^+_{k',s'}(0)\,a_{k'-q,s'}(0). \tag{12}$$

The term $\hat{R}_2^{(el)}$ describing the diagram with two electron-hole loops is equal to

$$\hat{R}_2^{(el)} = 4\int_{-\infty}^{0}\int_{-\infty}^{0}\int_{-\infty}^{0} dt_1dt_2dt_3$$
$$\times \sum_{kk'qq'ss'}\sum_{k_1}{}^{(q)}\sum_{k_2}{}^{(-q)}\frac{g^5(q)\,g(q')\sqrt{\omega^5(q)\,\omega(q')}}{(2V)^3}\,a^+_{k,s}(0)\,a_{k+q,s}(0)$$
$$\times a^+_{k's'}(t_3)\,a_{k'+q',s'}(t_3)\,\dot{B}_{q'}(t_2)\,\dot{B}_{q'}(t_3)\,C_{k_1,q}(0,t_1)\,C_{k_2,-q}(t_1,t_2)$$
$$\approx -\frac{8}{(2V)^3}\sum_{kk'qss'}\frac{g^6(q)\,\omega^2(q)}{-z^2(k',q)+\omega^2(q)}\left[\sum_{k_1}{}^{(q)}\frac{1}{\tilde{\varepsilon}(k_1)+\tilde{\varepsilon}(k_1-q)}\right]^2$$
$$\times a^+_{k,s}(0)\,a_{k+q,s}(0)\,a^+_{k',s'}(0)\,a_{k'-q,s'}(0). \tag{13}$$

Apart from the terms described by Eqs. (12) and (13), $\hat{R}_1^{(el)}$ and $\hat{R}_2^{(el)}$ contain a number of terms with higher powers of ω in the numerator. One notes easily that the subsequent terms in the series expansion of $\hat{R}^{(el)}$ must differ from the preceding ones by an increase in the power of the expression

$$2A(q) = \frac{2g^2(q)}{V} \sum_{k_1}^{(q)} \frac{1}{\tilde{\varepsilon}(k_1) + \tilde{\varepsilon}(k_1 - q)} \tag{14}$$

under the summation sign. The final expression for $\hat{R}^{(el)}$ is thus in the approximation (3) of the form

$$\hat{R}^{(el)} = -\frac{1}{2V} \sum_{kk'q'ss'} \frac{g^2(q)\,\omega^2(q)\,\Xi(A)}{-z^2(k',q) + \omega^2(q)}$$

$$\times a^+_{k,s}(0)\,a_{k+q,s}(0)\,a^+_{k',s'}(0)\,a_{k'-q,s'}(0), \tag{15}$$

$$\Xi(A) = 1 + A + 2A^2 + 4A^3 + \ldots$$

$$= (1 - A)/(1 - 2A). \tag{16}$$

The function $I(kk'q)$ from (2) is according to (15) and (16) determined by the formula

$$I(kk'q) = \frac{g^2(q)\,\omega^2(q)}{-[\varepsilon(k') - \varepsilon(k'-q)]^2 + \omega^2(q)} \frac{1 - A(q)}{1 - 2A(q)}. \tag{17}$$

One can similarly sum the diagrams with external phonon lines and renormalize the phonon energy (although the result needs more unwieldy calculations).

The calculations given here make it possible to establish an inequality which must be satisfied by the constant

$$2\max A = g^2(dn/dE)_{E=E_F} = \rho.$$

Indeed, the convergence of the series (16) requires that $\rho < 1$, which is the same as Wentzel's estimate.[3]

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[3] G. Wentzel, Phys. Rev. **83**, 168 (1951).

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