ON THE DOPPLER EFFECT IN AN ANISOTROPIC AND GYROTROPIC MEDIUM

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The Doppler effect is treated for motion of an oscillator along the axis of a gyrotropic anisotropic crystal. General formulas for the energy of the radiation are obtained; these can be used, in particular, to obtain formulas for the Cerenkov radiation of a charge and of a dipole. A number of properties of the radiation that are peculiar to an anisotropic, gyrotropic medium are investigated.

THE Doppler effect has great importance in many fields of application. Hitherto only the case of motion of a source in an isotropic medium has been studied; it has been shown that in the presence of dispersion, splitting of the Doppler frequency occurs (the complex Doppler effect[1,2]). In connection with numerous investigations of the ionosphere (for example, with the aid of high-speed earth satellites) and of an electronic plasma, which in the presence of a magnetic field has the properties of an anisotropic and gyrotropic crystal, it is of interest to investigate the properties of the field of a radiator moving in such a medium. In this case there arise new effects, manifesting themselves in changes of the components of the electromagnetic field and in an additional splitting of the original frequency, dependent on the degree of anisotropy and on the value of the gyration parameter.

In the present article, in order to elucidate the principal peculiarities of the phenomenon, we consider the problem of the field of an electric oscillator of arbitrary orientation, moving along the axis of an anisotropic and gyrotropic crystal.

1. Let a point oscillator be moving with constant velocity v along the z axis, which coincides with the crystal axis; let the oscillator have, in its proper system of coordinates, frequency ω_0', electric moment $\mathbf{p}_0' \cos \omega_0' t$, and magnetic moment $\mathbf{m}_0' \cos \omega_0' t$. Just as in reference 1, we replace the moving oscillator by a system of "equivalent" electric and magnetic harmonic oscillators of density

$$\mathbf{p}_\omega = \frac{\mathbf{p}_0}{2\pi v} \cos\left(\frac{\omega_0}{v} z\right) e^{-i\omega z/v} \delta(x)\,\delta(y),$$

$$\mathbf{m}_\omega = \frac{\mathbf{m}_0}{2\pi v} \cos\left(\frac{\omega_0}{v} z\right) e^{-i\omega z/v} \delta(x)\delta(y), \qquad (1.1)$$

where $\omega_0 = \sqrt{1-\beta^2}\,\omega_0'$ is the frequency measured in a stationary system of coordinates, and $\mathbf{p}_0$ and $\mathbf{m}_0$ are the electric and magnetic moments of the oscillator in the stationary system of coordinates; they are related to $\mathbf{p}_0'$ and $\mathbf{m}_0'$ by the appropriate relativistic transformation formulas.[1*] The currents and charges at each point of space are determined by the formulas

$$i\omega \mathbf{p}_\omega + c \,\mathrm{curl}\, \mathbf{m}_\omega = \mathbf{j}_\omega, \qquad \mathrm{div}\, \mathbf{p}_\omega = -\rho_\omega \qquad (1.2)$$

Thus the problem reduces to that of finding the electromagnetic field in the medium under consideration for given sources, when their density and therefore the field vectors depend on z through a factor $\exp[-i(\omega \pm \omega_0)z/v]$.

Maxwell's equations for the Fourier components of the field in an anisotropic and gyrotropic medium have the form

$$\mathrm{curl}\,\mathbf{H}_\omega = ik\hat{\varepsilon}\mathbf{E}_\omega + 4\pi\mathbf{j}_\omega/c, \qquad \mathrm{div}\,(\hat{\varepsilon}\mathbf{E}_\omega) = 4\pi\rho_\omega,$$
$$\mathrm{curl}\,\mathbf{E}_\omega = -ik\mathbf{H}_\omega, \qquad \mathrm{div}\,\mathbf{H}_\omega = 0, \qquad (1.3)$$

where

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon_1 & -ig & 0 \\ ig & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \end{pmatrix}.$$

The magnetic permeability of the medium is taken equal to unity.

It can be shown that the solution of the system (1.3) can be expressed in terms of two scalar functions ψ_1 and ψ_2, which play the role of potentials for (1.3), and which satisfy the equation

$$\Delta_2\psi_{1,2} + \sigma_{1,2}^2\psi_{1,2} = 4\pi f_{1,2}, \qquad (1.4)$$

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*We mention that the relativistic electric moment of a polarizable circuit, with a magnetic moment oriented perpendicular to the velocity, is to be calculated by the formula $\mathbf{p}_0 = (\hat{\varepsilon}_1/c)[\mathbf{v} \times \mathbf{m}_0']$, where $\hat{\varepsilon}_1$ is the dielectric tensor of the material of the circuit (as is done in the work of Ginzburg and Éĭdman[8]).

where $\sigma_{1,2}$ are the roots of the equation

$$\sigma^4 - \left[\left(1 + \frac{\varepsilon_1}{\varepsilon_2}\right) s^2 - \frac{k^2 g^2}{\varepsilon_1}\right]\sigma^2 + \frac{\varepsilon_1}{\varepsilon_2} s^4 - \frac{k^4 g^2 \varepsilon_2}{\varepsilon_1} = 0, \quad (1.5)$$

and where

$$f_{1,2} = \frac{i\sigma_{1,2}^2}{\varepsilon_2 \omega} j_{\omega z} - \frac{\sigma_{1,2}^2 - k^2\varepsilon_2}{\varepsilon_2 \mu \omega} \operatorname{div}_2 \mathbf{j}_\omega - \frac{i(\sigma_{1,2}^2 - s^2)\,\varepsilon_1}{\varepsilon_2 \mu \omega g} \operatorname{curl}_z \mathbf{j}_\omega,$$

$$s^2 = \varepsilon_2 (k^2 \varepsilon_1 - \mu^2) / \varepsilon_1, \qquad \mu = (\omega \pm \omega_0) / v,$$

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \operatorname{div}_2 \mathbf{j}_\omega = \partial j_{\omega x} / \partial x + \partial j_{\omega y} / \partial y. \quad (1.6)$$

The field vectors are found from the relations

$$\mathbf{E}_{\omega t} = \hat{e}_1 \nabla \psi_1 - \hat{e}_2 \nabla \psi_2 + \frac{4\pi}{c} \hat{p} \mathbf{j}_\omega,$$

$$E_{\omega z} = [(\sigma_2^2 - s^2)\psi_1 - (\sigma_1^2 - s^2)\psi_2] / (\sigma_2^2 - \sigma_1^2),$$

$$\mathbf{H}_\omega = (i/k) \operatorname{curl} \mathbf{E}_\omega, \quad (1.7)$$

where $\mathbf{E}_{\omega t}$ is the transverse component of the field, and where $\hat{e}_{1,2}$ and $\hat{p}$ are defined as operators:

$$\hat{e}_n = \begin{pmatrix} ie_{n1} & e_{n2} \\ -e_{n2} & ie_{n1} \end{pmatrix}, \qquad \hat{p} = \begin{pmatrix} ip_1 & -p_2 \\ p_2 & ip_1 \end{pmatrix},$$

$$e_{n1} = \mu\varepsilon_2 (\sigma_n^2 - \varepsilon_1 s^2/\varepsilon_2) / \varepsilon_1 \sigma_n^2 (\sigma_2^2 - \sigma_1^2),$$

$$e_{n2} = k^2 \varepsilon_2 \mu g / \sigma_n^2 \varepsilon_1 (\sigma_2^2 - \sigma_1^2),$$

$$p_1 = ks^2 / \sigma_1^2 \sigma_2^2, \qquad p_2 = gk^3 \varepsilon_2 / \varepsilon_1 \sigma_1^2 \sigma_2^2, \qquad n = 1.2. \quad (1.8)$$

We mention that the proposed method, apart from the case considered here, may prove useful for solution of the problem of excitation of a gyrotropic medium by an arbitrary current.

2. Of greatest interest are two physically distinct orientations of the oscillator: parallel and perpendicular to the z axis. For simplicity we restrict ourselves to these two cases; the solution in the case of arbitrary orientation of the oscillator can be obtained without difficulty from our formulas.

a) Electric oscillator parallel to the z axis. The functions ψ_1 and ψ_2 are found from the equations

$$\nabla_2^2 \psi_{1,2} + \sigma_{1,2}^2 \psi_{1,2} = -(\sigma_{1,2}^2 p_0 / \varepsilon_2 v)\, e^{-i(\omega \pm \omega_0) z/v} \delta(x)\, \delta(y), \quad (2.1)$$

where

$$p_0 = p_0' \sqrt{1 - \beta^2}.$$

The solution of (2.1) has the form

$$\psi_{1,2} = (\sigma_{1,2}^2 p_0 / 4v\varepsilon_2)\, b_{1,2}(r)\, e^{-i(\omega \pm \omega_0) z/v}, \quad (2.2)$$

where

$$b_{1,2}(r) = \begin{cases} -iH_0^{(2)}(\sigma_{1,2} r), & \omega > 0, \\ iH_0^{(1)}(\sigma_{1,2} r), & \omega < 0, \end{cases} \quad (2.3)$$

and where $H_0^{(1)}$ and $H_0^{(2)}$ are the Hankel functions of the first and second kinds. In this case formulas (2.2), (1.7), and (1.8) fully determine the field.

b) Electric oscillator perpendicular to the z axis.

Let the moment of the oscillator be oriented along the x axis. In this case the oscillator acquires a magnetic moment directed along the y axis and equal to $m_y = -\beta p_0'$. On carrying out a calculation similar to the preceding one, we find from (1.4) for this case

$$\psi_{1,2} = \frac{p_0'}{4\omega\varepsilon_2(\omega \pm \omega_0)} \left[i(\sigma_{1,2}^2 \pm k^2 \varepsilon_2 \omega_0) \cos\varphi \mp \right.$$

$$\left. \mp \frac{\omega_0 \varepsilon_1}{g} (\sigma_{1,2}^2 - s^2) \sin\varphi \right] \frac{\partial b_{1,2}}{\partial r} e^{-i(\omega \pm \omega) z/v}, \quad (2.4)$$

where φ is a polar angle measured from the x axis. The field vectors can be easily found with the aid of (2.4) and (1.7). Expressions for the components of the field of a magnetic oscillator are found in a completely analogous manner. We remark that the formulas obtained possess great generality. For example, the components of the Cerenkov radiation field of a dipole with moment π_0' can be obtained without difficulty from our formulas by setting $\omega_0 = 0$ and $p_0' = \pi_0'$; the components of the Cerenkov radiation field of an electron, by setting $\omega_0 = 0$ and $p_0 = -ive/\omega$ (see reference 3).

3. In the wave zone, by use of the asymptotic formulas for the Hankel functions, the components of the field can be expressed in the form

$$a \exp\left[-i\left(\frac{\omega \pm \omega_0}{v} z + \sigma_{1,2} r - \omega t\right)\right]. \quad (3.1)$$

If we require that the expressions for the field components have the form of a traveling plane wave with wave normal oriented at angle θ to the axis, i.e., that

$$\omega n \cos\theta / c = (\omega \pm \omega_0) / v, \qquad \omega n \sin\theta / c = \sigma_{1,2}, \quad (3.2)$$

then it is not difficult to obtain the condition for the Doppler frequency,

$$\omega = \omega_0 / | 1 - \beta n(\omega, \theta) \cos\theta |, \quad (3.3)$$

which agrees with the usual Doppler condition for an isotropic medium; but the refractive index $n(\omega, \theta)$ for the medium under consideration is a complicated function of ω and θ and is different for waves of different polarizations. By substituting (3.2) in (1.5) and solving the biquadratic equation, one can obtain the usual formula for the index of refraction of a gyrotropic medium. Certain results of Frank[1] are also easily generalized to a gyrotropic medium; for instance, the angular width of a spectral line is determined by the formula

$$\Delta\theta = \frac{\pm 2\pi (n_1^2 - n_2^2)(\varepsilon_1 \sin^2\theta + \varepsilon_2 \cos^2\theta)}{T\beta n_{1,2} \sin\theta [\varepsilon_1^2 - g^2 - \varepsilon_1\varepsilon_2 - n_{1,2}^2 (\varepsilon_2 \cos^2\theta + \varepsilon_1 (1 + \sin^2\theta))]}, \qquad (3.4)$$

where T is the travel time of the oscillator.

As in an isotropic medium, at oscillator speeds above that of light and at not too large angles, splitting of the Doppler line occurs (the complex Doppler effect). A sufficient condition for this effect is (cf. reference 1)

$$\beta n(\omega, \theta) \cos\theta > 1. \qquad (3.5)$$

An estimate of the derivative of the left side of (3.5) shows that the point $\theta = 0$ is a maximum, i.e., a necessary condition for the complex effect at superluminal oscillator speeds can be written in the form

$$(\varepsilon_1 \pm g)\beta^2 > 1. \qquad (3.6)$$

For a plasma in a magnetic field,

$$\varepsilon_1 = 1 - \frac{\omega_n^2}{\omega^2 - \omega_H^2}, \quad \varepsilon_2 = 1 - \frac{\omega_n^2}{\omega^2},$$

$$g = \frac{\omega_n^2 \omega_H}{\omega(\omega^2 - \omega_H^2)} \qquad (3.7)$$

(where $\omega_n = \sqrt{4\pi N e^2/m}$ and $\omega_H = eH/mc$ are respectively the plasma and gyromagnetic frequencies), and (3.6) takes the form

$$(1 - \omega_n^2 / \omega(\omega \pm \omega_H))\beta^2 > 1. \qquad (3.8)$$

It is not difficult to see that (3.8) can hold only when $\omega < \omega_H$ for the extraordinary wave.*

The condition (3.5) is by no means a necessary condition for the complex Doppler effect. In general the effect is possible at speeds much less than the speed of light in the medium, if only the following condition is satisfied (cf. reference 1):

$$\omega_0\beta\,[\partial n / \partial\omega]_{\omega=\omega_\theta} \cos\theta > 1. \qquad (3.9)$$

As in the case of the superluminal complex effect, a necessary condition for fulfilment of (3.9) is

$$\omega_0\beta\left[\frac{d}{d\omega}\sqrt{\varepsilon_1 \pm g}\right]_{\omega=\omega_\theta} > 1. \qquad (3.10)$$

It has been shown[4] that the complex effect can occur in a plasma in a magnetic field either at frequencies near the gyromagnetic frequency of the plasma, or under the condition $\epsilon_1 \pm g \approx 0$.

4. We consider the energy radiated by a moving oscillator for two different orientations of it. The amount of energy emitted by the oscillator per unit length of its path we determine with the aid of Poynting's theorem, using the asymptotic expressions for the cylinder functions. After somewhat laborious transformations we obtain, for an oscillator oriented along the velocity,

$$\frac{d\mathcal{E}}{dz} = \frac{p_0^2}{4v^2}\left(\int \frac{\sigma_1^2 (s^2 - \sigma_2^2)}{\varepsilon_2 (\sigma_2^2 - \sigma_1^2)}\,\omega\, d\omega + \int \frac{\sigma_2^2 (s^2 - \sigma_1^2)}{\varepsilon_2 (\sigma_1^2 - \sigma_2^2)}\,\omega\, d\omega\right), \qquad (4.1)$$

where the first and second integrals extend over the frequency ranges $\sigma_1^2(\pm\omega_0) > 0$ and $\sigma_2^2(\pm\omega_0) > 0$, respectively, and $\omega > 0$. The sign in $\pm\omega_0$ means that the parentheses in (4.1) actually contain a sum of four terms; $+\omega_0$ occurs in the first two and $-\omega_0$ in the second two. We remark that the terms with $+\omega_0$ differ from zero only at superluminal oscillator speeds.

A similar calculation gives the amount of energy emitted by an oscillator with moment oriented perpendicular to the velocity; in this case it is possible to obtain the distribution of energy density with angle φ. After a series of laborious calculations we get

$$\frac{d\mathcal{E}(\varphi)}{dz} = \frac{p_0'^2}{8\pi v^2}\Bigg[\int\limits_{\sigma_1^2(\pm\omega_0)>0} \frac{[g^2(\sigma_1^2\omega \pm k^2\varepsilon_2\omega_0)^2\cos^2\varphi - \varepsilon_1^2\omega_0^2(\sigma_1^2 - s^2)^2\sin^2\varphi](s^2 - \sigma_2^2)}{g^2\varepsilon_2\omega(\omega \pm \omega_0)^2(\sigma_2^2 - \sigma_1^2)}\,d\omega$$
$$+ \int\limits_{\sigma_2^2(\pm\omega_0)>0} \frac{[g^2(\sigma_2^2\omega \pm k^2\varepsilon_2\omega_0)^2\cos^2\varphi - \varepsilon_1^2\omega_0^2(\sigma_2^2 - s^2)^2\sin^2\varphi](s^2 - \sigma_1^2)}{g^2\varepsilon_2\omega(\omega \pm \omega_0)^2(\sigma_1^2 - \sigma_2^2)}\,d\omega\Bigg]. \qquad (4.2)$$

The total energy can be obtained without difficulty by integration over φ.

We note the great generality of the formulas obtained. In fact, by setting $\omega_0 = 0$ and $p_0^2/2 = \pi_0^2$ we obtain a formula for the energy of the Cerenkov radiation of a constant dipole with moment π_0 in the proper coordinate system:

$$\frac{d\mathcal{E}}{dz} = \frac{\pi_0'^2(1-\beta^2)}{v^4}\Bigg[\int\limits_{\varkappa_1^2>0} \frac{\varkappa_1^2(\varepsilon_2(\varepsilon_1\beta^2 - 1) - \varepsilon_1\varkappa_2^2)}{\varepsilon_1\varepsilon_2(\varkappa_2^2 - \varkappa_1^2)}\,\omega^3 d\omega$$
$$+ \int\limits_{\varkappa_2^2>0} \frac{\varkappa_2^2(\varepsilon_2(\varepsilon_1\beta^2 - 1) - \varepsilon_1\varkappa_1^2)}{\varepsilon_1\varepsilon_2(\varkappa_1^2 - \varkappa_2^2)}\,\omega^3 d\omega\Bigg],$$

$$\sigma_{1,2}^2 = \frac{\omega^2}{v^2}\varkappa_{1,2}^2 = \frac{\omega^2}{2v^2}\Bigg((\varepsilon_1\beta^2 - 1)\left(1 + \frac{\varepsilon_2}{\varepsilon_1}\right) - \frac{g^2\beta^2}{\varepsilon_1}$$
$$\pm\left[\left[(\varepsilon_1\beta^2 - 1)\left(\frac{\varepsilon_2}{\varepsilon_1} - 1\right) + \frac{g^2\beta^2}{\varepsilon_1}\right]^2 + 4\frac{g^2\beta^2}{\varepsilon_1}\right]^{1/2}\Bigg) \qquad (4.3)$$

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*By extraordinary and ordinary waves will be understood those waves that reduce for $g = 0$ to the corresponding waves in a uniaxial crystal.

for a dipole oriented parallel to the velocity, and

$$\frac{d\mathcal{E}(\varphi)}{dz} = \frac{\pi_0'^2 \cos^2\varphi}{2\pi v^4}\Big[\int\limits_{\varkappa_1^2>0} \frac{\varkappa_1^4(\varepsilon_2(\varepsilon_1\beta^2-1)-\varepsilon_1\varkappa_2^2)}{\varepsilon_1\varepsilon_2(\varkappa_2^2-\varkappa_1^2)}\omega^3\,d\omega + \int\limits_{\varkappa_2^2>0} \frac{\varkappa_2^4(\varepsilon_2(\varepsilon_1\beta^2-1)-\varepsilon_1\varkappa_1^2)}{\varepsilon_1\varepsilon_2(\varkappa_1^2-\varkappa_2^2)}\omega^3\,d\omega\Big] \quad (4.4)$$

for a dipole oriented perpendicular to the velocity. And finally, by setting $\omega_0 = 0$ and $p_0' = -ive/\omega$ and substituting in (4.1), a formula is automatically obtained for the energy of a Cerenkov electron flying at superluminal speed along the z axis (cf. reference 3).

5. By way of illustration, we consider a uniaxial crystal. On setting $g = 0$ in (4.1) and (4.2), we get

$$\frac{d\mathcal{E}_\parallel}{dz} = \frac{p_0^2}{4c^2v^2}\int\limits_{s^2(\pm\omega_0)>0}\left(\omega^2 - \frac{(\omega\pm\omega_0)^2}{\varepsilon_1\beta^2}\right)\omega\,d\omega \quad (5.1)$$

for an oscillator parallel to the velocity, and

$$\frac{d\mathcal{E}_\perp}{dz} = \frac{p_0'^2}{8c^4}\Big[\int\limits_{s^2(\pm\omega_0)>0}\varepsilon_2\left(\omega - \frac{\omega\pm\omega_0}{\varepsilon_1\beta^2}\right)\omega\,d\omega + \frac{\omega_0}{\beta^2}\int\limits_{(\varepsilon_1/\varepsilon_2)s^2(\pm\omega_0)>0}\omega\,d\omega\Big] \quad (5.2)$$

for an oscillator perpendicular to the velocity.

We notice that for $\omega_0 = 0$, $\epsilon_1 = \epsilon_2$, and $p_0^2/2 = \pi_0^2$, formulas for the energy of Cerenkov radiation are obtained that agree with those obtained by Frank.[5]

If we neglect dispersion — that is, if the constants of the medium change little in the range of the Doppler frequencies emitted by the source — then it is possible to carry out the integration in (5.1) and (5.2). After simple calculations, (5.1) and (5.2) take the form, for $\beta\sqrt{\epsilon_1} < 1$,

$$d\mathcal{E}_\parallel / dz = \omega_0'^4 p_0'^2 \sqrt{\varepsilon_1}(1-\beta^2)^3 / 3c^3v(1-\beta^2\varepsilon_1)^3, \quad (5.3)$$

$$d\mathcal{E}_\perp / dz = \omega_0'^4 p_0'^2(\varepsilon_2 + 3\varepsilon_1)/12c^3v\sqrt{\varepsilon_1}(1-\beta^2\varepsilon_1)^2. \quad (5.4)$$

For $\beta\sqrt{\epsilon_1} > 1$, the integrals in (5.1) and (5.2) diverge; this is connected with the fact that in this case it is not permissible to neglect dispersion. We note that (5.3) indicates that the total radiation energy is independent of ϵ_2. Comparison of (5.3) and (5.4) also shows an essential dependence of the radiated energy on the orientation of the oscillator. For $\epsilon_1 = \epsilon_2$, (5.3) and (5.4) reduce to the corresponding formulas for the radiation energy of an oscillator in an isotropic medium (cf. reference 1).

In a medium without dispersion, the frequency radiated by the oscillator is simply related to the angle at which this frequency is observed. That is, if in (5.1) and (5.2) we transform from integration over ω to integration over angles with the aid of

$$\omega_{1,2} = \omega_0 / |1 - \beta n_{1,2}(\theta)\cos\theta|, \quad (5.5)$$

where

$$n_1(\theta) = \sqrt{\varepsilon_1}; \qquad 1/n_2^2(\theta) = \cos^2\theta/\varepsilon_1 + \sin^2\theta/\varepsilon_2,$$

and if we consider the path of the oscillator to be finite and of length l (cf. reference 2), then we can obtain a formula for the distribution of the radiation energy with angle θ. After simple transformations it follows from (5.1) and (5.2) that

$$W_\parallel(\theta) = p_0^2 l\omega_0^4 n_2^5(\theta)\sin^2\theta / 8c^3v\varepsilon_2^2 |1 - n_2(\theta)\beta\cos\theta|^5 \quad (5.6)$$

for an oscillator oriented along the velocity, and

$$W_\perp(\theta,\varphi) = \frac{p_0'^2\omega_0^4 l}{8c^3v}\left[\frac{(\varepsilon_1\beta - n_2(\theta)\cos\theta)^2 n_2^3(\theta)\cos^2\varphi}{\varepsilon_1^2|1-\beta n_2(\theta)\cos\theta|^5} + \frac{\sqrt{\varepsilon_1}\sin^2\varphi}{|1-\beta\sqrt{\varepsilon_1}\cos\theta|^3}\right] \quad (5.7)$$

for an oscillator oriented perpendicular to the velocity. For $\epsilon_1 = \epsilon_2$, (5.6) and (5.7) reduce to the analogous formulas for an isotropic medium (cf. reference 1).

6. The Doppler effect in a uniaxial gyrotropic crystal has a number of peculiarities as compared with an isotropic medium. As was mentioned by Pafomov,[6] in the range of frequencies in which the radial group velocity is negative (i.e., the projections of the group velocity and of the wave vector along the radius have opposite signs), in order that the solution shall have physical meaning it is necessary to use advanced potentials. If in this case $\cos\theta$ and $\cos\varphi$ (φ is the angle between the z axis and the direction of the beam) have the same sign, there arise a peculiar "inverted" Doppler effect — that is, lower frequencies are radiated forward and higher frequencies backward. This same effect occurs if $W_r > 0$ and if $\cos\theta$ and $\cos\varphi$ have opposite signs.

The projection W_r of the group velocity along the radius, in a gyrotropic medium, has the form[7]

$$W_r = -\frac{\partial n\cos\theta/\partial\theta}{(n/c)\,\partial\omega n/\partial\omega} = \frac{c\,[n^2(\varepsilon_2\cos^2\theta + \varepsilon_1(1+\sin^2\theta)) - \varepsilon_1^2 + g^2 - \varepsilon_1\varepsilon_2]\sin\theta}{\frac{\partial(\omega n)}{\partial\omega}[2n^2(\varepsilon_1\sin^2\theta + \varepsilon_2\cos^2\theta) - (\varepsilon_1 - g^2)\sin^2\theta - \varepsilon_1\varepsilon_2(1+\cos^2\theta)]}, \quad (6.2)$$

and the angles θ and φ are connected by the relation

$$\tan\varphi = -\frac{\partial n\cos\theta/\partial\theta}{\partial n\sin\theta/\partial\theta}$$

$$= \frac{[n^2(\varepsilon_2\cos^2\theta + \varepsilon_1(1+\sin^2\theta)) - \varepsilon_1^2 + g^2 - \varepsilon_1\varepsilon_2]\tan\theta}{n^2(\varepsilon_1\sin^2\theta + \varepsilon_2\cos^2\theta + \varepsilon_2) - 2\varepsilon_1\varepsilon_2}. \quad (6.3)$$

As was mentioned in reference 4, in a plasma with a magnetic field at angles θ close to $\pi/2$, in the range of frequencies

$$[{}^1/_2(\omega_H^2 + \sqrt{4\omega_n^4 + \omega_H^4})]^{1/2} < \omega < \sqrt{\omega_n^2 + \omega_H^2} \quad (6.4)$$

an "inverted" Doppler effect occurs. For a more general conclusion a more detailed study of formulas (6.2) and (6.3) is required.

We point out, furthermore, that in a uniaxial anisotropic crystal without optical activity, if the crystal constants are determined from an oscillator model (as was done in reference 6), there exists a frequency range in which an "inverted" Doppler effect can occur at any angles.

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