STABILITY OF AN IDEALLY CONDUCTING LIQUID FLOWING BETWEEN CYLINDERS ROTATING IN A MAGNETIC FIELD

E. P. VELIKHOV

Moscow State University

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Sufficient conditions for the stability of an ideal liquid flowing in axial and toroidal magnetic fields are derived. Critical values of the magnetic fields which stabilize the flow are obtained and a physical interpretation of the results is presented.

1. INTRODUCTION

RAYLEIGH,[1] Taylor,[2] Meksyn,[3] and Synge[4] investigated the stability of flow of a viscous, incompressible liquid between rotating cylinders. The classical condition of stability (see Landau and Lifshitz[5])

$$\Omega_1 R_1^2 < \Omega_2 R_2^2 \tag{1.1}$$

follows from the conservation of angular momentum of a liquid particle (Ω_1, Ω_2 are the angular velocities and R_1, R_2 are the radii of the rotating cylinders).

We examine in this paper the stability of flow of an ideally conducting nonviscous liquid in a magnetic field. Chandrasekhar[6] solved a similar problem for a poorly-conducting viscous liquid.

The motion of the liquid and the field in it are described by the equations of magnetic hydrodynamics for an ideally conducting liquid:

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla\Phi + \frac{1}{4\pi}(\mathbf{B}\nabla)\mathbf{B} + \nu\nabla^2\mathbf{V}, \quad \operatorname{div}\mathbf{V} = 0,$$
$$\frac{d\mathbf{B}}{dt} = (\mathbf{B}\nabla)\mathbf{V}, \quad \operatorname{div}\mathbf{B} = 0 \tag{1.2}$$

(V is the velocity, B is the induction of the magnetic field, and Φ is the total pressure of the substance and the field).

Independent of the magnitude of the viscosity, these equations admit of the following stationary solution:

$$\mathbf{V}_0 = (ar + e/r)\,\mathbf{e}_\varphi, \quad \mathbf{B}_0 = \mathbf{B}_{0\varphi}(r)\,\mathbf{e}_\varphi + \mathbf{B}_{0z}\mathbf{e}_z,$$
$$a = -(\Omega_1 R_1^2 - \Omega_2 R_2^2)/(R_2^2 - R_1^2),$$
$$e = (\Omega_1 - \Omega_2) R_1^2 R_2^2/(R_2^2 - R_1^2). \tag{1.3}$$

In the following discussion we ignore the effect of viscosity on perturbations of the stationary flow.

We investigate the "linear" stability of flow, i.e., the stability relative to disturbances of infinitesimal amplitude. The system (1.2), after substituting the disturbed values of velocity, field, and pressure

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{v}, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad \Phi = \Phi_0 + \phi, \tag{1.4}$$

neglecting viscosity and linearization, becomes a system of linear differential equations with coefficients that are independent of z, φt in cylindrical coordinates. Solutions of such a system have the form

$$\mathbf{v} = \mathbf{v}(r)\exp\{i(m\varphi + kz + \omega t)\},$$
$$\mathbf{b} = \mathbf{b}(r)\exp\{i(m\varphi + kz + \omega t)\},$$
$$\phi = \phi(r)\exp\{i(m\varphi + kz + \omega t)\}. \tag{1.5}$$

Only disturbances of definite ω satisfy the homogeneous boundary conditions $v_r = 0$, $b_r = 0$ for $r = R_1, R_2$. The sign of the imaginary part $\operatorname{Im}\omega = \omega_1$ determines the stability of flow.

2. STABILITY OF FLOW IN AN AXIAL MAGNETIC FIELD

As can be easily shown, the most dangerous disturbances are the ones that do not depend on φ. The amplitudes of velocity and field disturbances are expressed in terms of the amplitude of the radial component of disturbance of the magnetic field in the following manner

$$v_r = (\omega/k)\, b_r/B_{0z}; \tag{2.1}$$

$$v_\varphi = \frac{2i}{k(\omega^2/k^2 - B_{0z}^2/4\pi\rho)}\left(\frac{\omega^2}{k^2}a + \frac{B_{0z}^2}{4\pi\rho}\frac{e}{r^2}\right)\frac{b_r}{B_{0z}}; \tag{2.2}$$

$$v_z = (i\omega/k^2 B_{0z})(db_r/dr + b_r/r); \tag{2.3}$$

$$b_\varphi = \frac{2i\omega}{k^2}\,\frac{a + e/r^2}{B_{0z}^2/4\pi\rho - \omega^2/k^2}\, b_r; \tag{2.4}$$

$$b_z = (i/k)(db_r/dr + b_r/r), \tag{2.5}$$

while the amplitude b_r itself is found from the equation

$$\left(\frac{B_{0z}^2}{4\pi\rho}-\frac{\omega^2}{k^2}\right)^2\left\{\frac{d}{dr}\,r\,\frac{d}{dr}-\frac{1}{r}-rk^2\right\}b_r+4\frac{\omega^2}{k^2}a\left(ar+\frac{e}{r}\right)b_r$$
$$+\frac{B_{0z}^2}{\pi\rho}\frac{e}{r^2}\left(ar+\frac{e}{r}\right)b_r=0 \qquad (2.6)$$

with boundary conditions

$$b_r=0 \quad \text{for } r=R_1,R_2. \qquad (2.7)$$

We shall show that ω^2 is a real number; i.e., there is possible either an aperiodic growth in the disturbances or an oscillation about the equilibrium point. Actually, multiplying Eq. (2.6) by b_r^*, the complex conjugate of b_r, we obtain

$$J=\left(\frac{B_{0z}^2}{4\pi\rho}-\frac{\omega^2}{k^2}\right)^2 J_1-4\frac{\omega^2}{k^2}aJ_2-\frac{B_{0z}^2}{\pi\rho}eJ_3=0, \qquad (2.8)$$

where

$$J_1=\int_{R_1}^{R_1}\left(r\,|b_r'|^2+\frac{1}{r}|b_r|^2+rk^2|b_r|^2\right)dr>0; \qquad (2.9)$$

$$J_2=\int_{R_1}^{R_2}V_{0\varphi}(r)\,|b_r|^2\,dr; \qquad (2.10)$$

$$J_3=\int_{R_1}^{R_2}V_{0\varphi}(r)\,|b_r|^2\frac{dr}{r^2}. \qquad (2.11)$$

The imaginary part of (2.8) is

$$\operatorname{Im}J=\frac{4\omega_i\omega_r}{k^2}\left\{\left(\frac{B_{0z}^2}{4\pi\rho}-\frac{\omega_r^2}{k^2}+\frac{\omega_i^2}{k^2}\right)J_1+2aJ_2\right\}=0. \qquad (2.12)$$

It is easy to see that the expression in braces is not equal to zero, for otherwise the real part of (2.8) would not vanish:

$$\operatorname{Re}J=-\left\{\left[\left(\frac{B_{0z}^2}{4\pi\rho}-\frac{\omega_r^2}{k^2}+\frac{\omega_i^2}{k^2}\right)^2\right.\right.$$
$$\left.\left.+\frac{4\omega_i^2\omega_r^2}{k^4}\right]J_1+\frac{B_{0z}^2}{\pi\rho}J_4\right\}<0, \qquad (2.13)$$

where

$$J_4=\int_{R_1}^{R_2}V_{0\varphi}^2\,|b_r|^2\frac{dr}{r}>0,$$

which contradicts Eq. (2.8). Therefore $\omega_1\omega_r=0$.

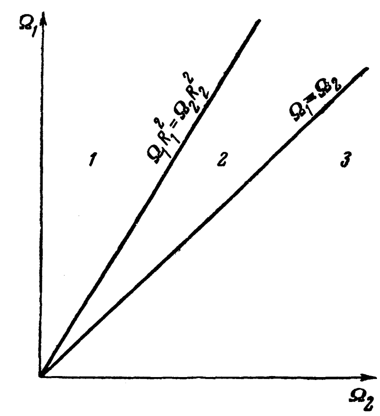
A sufficient condition for stability follows from expression (2.8). If the cylinders are rotating in the same direction, then $J_2>0$ and $J_3>0$, because $V_{0\varphi}>0$. Therefore when $e<0$ (we know that $a>0$) the flow is stable. The sufficient condition for stability has the form:

$$\Omega_2-\Omega_1>0. \qquad (2.14)$$

The same condition can be obtained more obviously by the Rayleigh method. Let us examine an element of liquid rotating in some layer with angular velocity Ω. If the element departs from the layer, then because of the ideal conductivity it will drag away the line of force to which it is "glued." Almost all of the line of force will remain in the original layer. Therefore it rotates with the former angular velocity and drags the fluid element after it. As a result, the element retains its previous angular velocity. But without a field, the axial momentum component is preserved, because there are no tangential forces acting on the element. Formally, the preservation of angular velocity during slow motion of the particle follows directly from expressions (2.1) through (2.5).

The element experiences three radial forces in the new layer: the quasi-elastic force of the distorted magnetic line of force, the centrifugal force, and the equilibrium pressure gradient in the new layer. If $d\Omega^2/dr>0$, i.e., condition (2.14) is not fulfilled, the centrifugal force exceeds the equilibrium pressure gradient. Thus, $\Omega_1>\Omega_2$ is the necessary condition for the existence of the centrifugal effect. In a sufficiently weak field [see (2.8)] this effect leads to instability of flow.

Without a field, similar arguments (see reference 5) lead to the condition for the existence of the centrifugal effect $\Omega_1R_1^2>\Omega_2R_2^2$ (Rayleigh condition). However, when $B\to 0$ condition (2.14) does not transform into the Rayleigh condition (see the figure). A paradox arises upon going in the limit from $B\to 0$ to $B=0$. Actually, at any large but finite conductivity in a weak field, if the disturbance has a long enough development time [see (2.20)], the field has enough time to diffuse from the disturbance and the element no longer retains its angular velocity. In this sense the situation is similar to the well-known transition from a slightly viscous to a nonviscous liquid. For example, flow between flat plates is stable when $R_g=\infty$ and unstable at any large but finite Reynolds number R_g. This is connected with the fact that stability of flow results from the law of conservation of the curl of the velocity. No matter how small the viscosity, the curl of the veloc-



Stability region of the Couette flow with (3) and without (2+3) field.

ity has time to diffuse from the disturbances near the resonance layer (the "layer of internal friction"), where the phase velocity of the disturbances is equal to the velocity of flow. Therefore the mechanism of stabilization does not operate near this layer (see reference 8).

The neglect of viscosity and of ohmic resistance is justified consequently, by the smallness of characteristic times as compared with diffusion times, i.e., by the fact that the parameters

$$P_m = B_{0z}\sigma\sqrt{4\pi}L/c^2\sqrt{\rho}, \tag{2.15}$$

$$P_g = B_{0z}L/\nu\sqrt{4\pi\rho} \tag{2.16}$$

are large as compared to unity (L is the characteristic length).

Neutral disturbances, as seen from expressions (2.1) through (2.5), are barrel-shaped disturbances of the field, brought about by azimuthal current. With this, the liquid tends to rotate as a solid body, while the velocity gradient builds up in the region of the enhanced field.

In a sufficiently strong field the flow is stabilized by a large quasi-elastic force. The critical value of the field can be found easily for the particular case of a small space between the cylinders. If the cylinders are rotating in the same direction, the coefficients in Eq. (2.6) can be replaced by their average values, and the equation can be solved in trigonometric functions; but if the cylinders are rotating in opposite directions, we must account for the linear terms, and Eq. (2.6) will reduce to the Airy equation. In the first case we have for disturbances with wave number k

$$\omega^2 = k^2\left\{\frac{B_{0z}^2}{4\pi\rho} - \frac{2\Omega_0(\Omega_1-\Omega_2)\,dR_0}{\pi^2+k^2d^2}\right\}, \tag{2.17}$$

where, as above, $d = R_2 - R_1$, $R_0 = \frac{1}{2}(R_2 + R_1)$, and $\Omega_0 = \Omega(R_0)$. When $k^2 \to 0$ we have $\omega^2 \to 0$, because long-wave disturbances interact too weakly with the field. When $k^2 \to k^2_{max}$ we have $\omega^2 \to 0$, because the short-wave disturbances distort the lines of force too much. From (2.17) it follows that to suppress the instability it is sufficient that

$$B_{0z}^2/8\pi > \rho\Omega_0(\Omega_1-\Omega_2)R_0d/\pi^2. \tag{2.18}$$

The wave number of dangerous disturbances is bounded from above,

$$k^2 \leqslant k^2_{max} = (\Lambda - \pi^2)/d^2, \tag{2.19}$$

as is the increment

$$\omega_i < \omega_{imax} = (B_{0z}\sqrt{\pi}/2\sqrt{\rho}\,d)(\sqrt{\Lambda}/\pi - 1), \tag{2.20}$$

while

$$\Lambda = 8\pi\rho\,\Omega_0(\Omega_1-\Omega_2)R_0d/B_{0z}^2. \tag{2.21}$$

For cylinders rotating in opposite directions, the results do not change qualitatively.

The flow near a cylinder rotating in an unbounded medium is unstable in a sufficiently weak field, as follows from condition (2.14). However, the same can be shown by a direct calculation. By substituting $b_r = \varphi/\sqrt{r}$ we transform (2.6) into a Schrödinger-type equation

$$\frac{d^2\psi}{dr^2} + \left\{-k^2 - \frac{3}{4}\frac{1}{r^2} + \frac{B_{0r}^2}{\pi\rho}\frac{(\Omega_1R_1)^2}{r^4(B_{0z}^2/4\pi\rho - \omega^2/k^2)^2}\right\}\psi = 0, \tag{2.22}$$

since $e = \Omega_1R_1^2$ and $a = 0$. The boundary conditions have the usual appearance: $\varphi = 0$ when $r = R_1$ and ∞. Therefore, if we have in the well, at $\omega^2 < 0$

$$U = \frac{3}{4}\frac{1}{r^2} + \frac{B_{0z}^2}{\pi\rho}\frac{\Omega_1^2R_1^4}{r^4(B_{0z}^2/4\pi\rho - \omega^2/k^2)^2} \text{ for } r > R_1,$$

$$U = \infty \text{ for } r < R_1,$$

and levels with $E = -k^2 < 0$ exist, the flow is unstable. In the quasi-classical approximation, the condition for the existence of a level has the form

$$\int_{R_1}^{R_0}\sqrt{-k^2-U}\,dr = \pi\left(n+\tfrac{3}{4}\right) \tag{2.23}$$

(see reference 9) where n is the number of zeros of the ψ functions in the well. For long waves, $k^2 \ll 1/R_0$, we obtain the condition:

$$\sqrt{R_0^2/R_1^2 - 1} + \sin^{-1}(R_1/R_0)$$

$$= (2\pi/\sqrt{3})(n + 3/4) + \pi/2, \tag{2.24}$$

where $$R_0 = \frac{2B_{0z}}{\sqrt{3\pi\rho}}\frac{\Omega_1R_1^2}{(B_{0z}^2/4\pi\rho - \omega^2/k^2)}.$$

To satisfy the quasi-classical approximation condition we set $n = 10$. Then

$$-\frac{\omega^2}{k^2} \approx \frac{B_{0z}}{\sqrt{4\pi\rho}}\left\{\frac{\Omega_1R_1}{18} - \frac{B_{0z}}{\sqrt{4\pi\rho}}\right\},$$

and in a field less than $\sqrt{4\pi\rho}\,\Omega_1R_1/18$ the flow is known to be unstable. The critical value of the field is obviously larger, but its determination requires an accurate solution of the problem.

Since the instability has a local character, a simple vortex is also unstable in an axial magnetic field of less than critical value. This instability can appear, for example, when a star rotates in its own magnetic field.

3. STABILITY OF FLOW IN AN AZIMUTHAL MAGNETIC FIELD

In the presence of a magnetic field directed along φ, ω^2 is determined as the eigenvalue of the differential equation for the radial component of velocity disturbance:

$$v_r'' + \frac{v_r'}{r} - \left\{k^2 + \frac{1}{r^2} - 4a\left(a + \frac{e}{r^2}\right)\frac{k^2}{\omega^2} + \frac{B_{0\varphi}}{2\pi\rho r}\frac{k^2}{\omega^2}\left(\frac{dB_{0\varphi}}{dr} - \frac{B_{0\varphi}}{r}\right)\right\} v_r = 0, \quad (3.1)$$

with boundary conditions

$$v_r = 0 \text{ for } r = R_1, R_2. \quad (3.2)$$

The problem is easily solved for a narrow space between cylinders rotating in the same direction, as in Sec. 2. With this

$$\omega^2 = \left[4a\,\Omega_0 - \frac{B_{0\varphi}}{2\pi\rho r}\left(\frac{dB_{0\varphi}}{dr} - \frac{B_{0\varphi}}{r}\right)\Big|_{r=R_0}\right]\frac{k^2d^2}{\pi^2 + k^2d^2}. \quad (3.3)$$

For stability it is necessary and sufficient that $\omega^2 > 0$. In the particular case of a power dependence of the field on the distance to the axis,

$$B_{0\varphi} = B_{0\varphi}(r/R_0)^n \qquad (n \ll R_0/d) \quad (3.4)$$

and relation (3.3) becomes

$$\omega^2 = \left[4a\Omega_0 - \frac{B_{0\varphi}^2(n-1)}{2\pi\rho R_0^2}\right]\frac{k^2d^2}{\pi^2 + k^2d^2}. \quad (3.5)$$

The faster the field drops off to the periphery, the greater the stabilizing effect of the field.

A more graphical way of obtaining the same result is by the Rayleigh method. An element rotating in some layer, is "glued" to the corresponding lines of force. If it leaves the layer, no force arises along φ. Therefore, as in the absence of a field, the axial component of the angular momentum of the element is conserved. However, the lines of force are distorted, and the entire force tube is dragged into motion along with the element. If the pressure gradient in the external layer cannot equalize the centrifugal force $\rho\Omega^2 r$, i.e., if $\Omega d(\Omega r^2)/dr < 0$, the force tube will be accelerated in the direction of motion. But the volume of the stretched tube is conserved, while its radius decreases as $1/r$. Because of the conservation of the magnetic flux, the field in the tube increases as r. If the equilibrium field grows faster, then the decrease in the strong field entering the tube from the periphery is not compensated by the growth of the weak field in the tube coming to replace it from the inside. Part of the magnetic energy changes into disturbance energy.

If the equilibrium field increases as r, then the transfer of the force tubes does not influence the development of the instability.

If the field decreases from the axis or grows slower than r then, for radial circulation, the perturbation energy changes into energy of the magnetic field. In a sufficiently strong field the rate at which the tube gains energy from the centrifugal force is less than the rate of growth of the magnetic energy. The tube will stop, having used up all of the priming supply. On the other hand, if this supply is sufficient for the tube to reach the outer wall, there will be no further growth in average magnetic energy. Therefore the flow is stable only in linear approximation, i.e., one possible stationary flow is separated from the other by a potential barrier.

In this manner, the decrease in the velocity and the growth of the field bring about instability. It is interesting to evaluate their respective influences. For this, following Shafranov,[7] let us examine a flow of the form

$$B_0 = \beta\sqrt{4\pi\rho}\,V_0. \quad (3.6)$$

From (3.3) it follows that

$$\omega^2 = 2(\Omega_1 - \Omega_2)V_0(\beta^2 - 1)k^2d/(\pi^2 + k^2d^2), \quad (3.7)$$

i.e., the flow is stable if $\Omega_1 > \Omega_2$, when $\beta^2 < 1$, and if $\Omega_1 < \Omega_2$ when $\beta^2 > 1$. $\beta^2 = 1$ is a critical value at which the effects of the field and the velocity are balanced.

In conclusion let us point out that conditions (2.15), (2.16), limiting the region of applicability of the results, are fulfilled both in experiments with certain liquid metals (Na) and under astrophysical conditions.

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