THE NUCLEAR INTERACTION IN THE SCATTERING OF CHARGED PARTICLES FROM NONSPHERICAL NUCLEI

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The nuclear interaction in the scattering of charged particles with energies close to the height of the Coulomb barrier from black, nonspherical nuclei is considered.

In a previous paper[1] the author obtained an expression for the wave function of a charged particle scattered from a nonspherical nucleus, containing the nuclear amplitudes $b^{\Omega}_{ll'}$. The aim of the present paper is to actually calculate these amplitudes for the case of a black nucleus.* This case has practical interest, since the condition of complete absorption of the incoming particles in the nucleus is, apparently, well satisfied for α particles (which are widely used in Coulomb excitation experiments) in the energy region under consideration ($E \approx 20$ Mev). The determination of the nuclear amplitudes (23.1) is in general very involved, since the potential, and therefore the wave functions, have quite different symmetries inside and outside the nucleus. The conditions for the joining of the wave function at the nuclear surface therefore lead to a complicated system of algebraic equations for the amplitudes $b^{\Omega}_{ll'}$. It is obvious that this system can be solved only numerically. However, this situation becomes very much simpler in the case of a black nucleus, since it is known[2] that in this case the wave function satisfies on the nuclear surface the condition

$$\frac{\partial}{\partial r}(r\Psi) = -iK(r\Psi), \qquad (1)$$

where K is a certain constant. With this boundary condition the amplitudes $b^{\Omega}_{ll'}$ can be found without considering the solution in the internal region at all.

1. BOUNDARY CONDITION FOR A DIFFUSE SURFACE

In writing down the boundary condition (1), where K is the complex wave vector of the particle inside the nucleus, we assumed that the nucleus has a well-defined boundary. In actual fact, the nuclear potential drops smoothly over a distance d which is of the order of magnitude of the range of the nuclear forces. Since $Kd \gtrsim 1$, this circumstance can have an appreciable effect on the scattering amplitude.

We are interested in the region of energies close to the height of the Coulomb barrier. To take account of the diffuseness of the boundary we can use the method proposed by Gribov.[3] He showed that, with an accuracy up to and including terms of order kd (where k is the wave number of the particle outside the nucleus),

$$d^2u/dr^2 = (2m/\hbar^2)Vu, \qquad (2)$$

in the diffuse surface region. Here u is the wave function multiplied by r, and V is the nuclear potential (the particle is assumed to be neutral). It is easily seen that the case of a charged particle is different from that considered in reference 3 mainly by the fact that the role of the wave number k is here taken over by the quantity

$$k' = \sqrt{2m|E-V_c|}/\hbar,$$

where V_c is the Coulomb potential. Up to terms of order $k'd$, the wave function therefore satisfies Eq. (2), as before.

We write the nuclear potential in the diffuse surface region in the form

$$V = \frac{\hbar^2}{2m}K^2 v\left[\frac{R_0(\theta)-r}{d}\right], \qquad (3)$$

where v is a dimensionless function with the property

$$v(x) \to 1, \quad x > 1; \qquad (4)$$
$$v(x) \to 0, \quad x < -1. \qquad (5)$$

At the internal end of the diffuse region we have again the old boundary condition (1). If, therefore, $\varphi_1[(R_0(\theta)-r)/d]$ is a solution of equation (2) of the form

$$\varphi_1 \sim e^{-iKr} \quad \text{for } v \sim 1, \qquad (6)$$
$$\varphi_1 \sim a + b[R_0(\theta)-r] \quad \text{for } v \sim 0, \qquad (7)$$

where a and b are constants uniquely determined

*We shall use the notations of reference 1. Reference to the formulas of this paper will be made in the form, e.g., (21.1).

by the form of the function v, then the wave function multiplied by r is, in the diffuse region,

$$u = A(\theta)\varphi_1. \quad (8)$$

For values of r corresponding to $v \approx 0$, this function must join smoothly to the wave function, multiplied by r, in the external region, which we denote by u^0. Expanding $u^0(r, \theta)$ in powers of $R_0(\theta) - r$ and discarding terms of order $(k'd)^2$, we obtain

$$u^0 \approx u^0(R_0(\theta), \theta)\left[1 - \frac{1}{u^0(R_0)}\frac{\partial u^0}{\partial r}\Big|_{r=R_0}(R_0(\theta) - r)\right]. \quad (9)$$

Comparing (9) and (7), we find the necessary conditions for equality of the two functions for $v \approx 0$:

$$\partial(r\Psi)/dr = -iK_{\text{eff}}(r\Psi) \quad \text{for } r = R_0(\theta), \quad (10)$$

where

$$K_{\text{eff}} = -ib/a. \quad (11)$$

The (complex) quantity b/a depends on the form of the function v. For example, in the case $v(x) = 1/(1 + e^{-x})$, which is important for applications,

$$K_{\text{eff}} = K \tanh \pi\gamma / \pi\gamma, \quad (12)$$

where $\gamma = Kd$. Hence the diffuseness of the boundary leads to the replacement of K by K_{eff} in the boundary condition (1).

2. THE SYSTEM OF EQUATIONS FOR THE AMPLITUDES $b_{ll'}^{\Omega}$

If the wave function Ψ is written in the form (24.1), the boundary condition (11) must be fulfilled for all functions (23.1).

We introduce the notations

$$dF_{l\Omega}(\rho)/d\rho = f_{l\Omega}(\rho)F_{l\Omega}(\rho),$$

$$dG_{l\Omega}(\rho)/d\rho = g_{l\Omega}(\rho)G_{l\Omega}(\rho), \quad (13)$$

$$\frac{F_{l\Omega}[\rho(\mu)]}{F_{l\Omega}} = \varphi_{l\Omega}(\mu), \quad \frac{G_{l\Omega}[\rho(\mu)]}{G_{l\Omega}} = \gamma_{l\Omega}(\mu), \quad (14)$$

$$G_{l\Omega}(\rho) = H_{l\Omega}(\rho) - iF_{e\Omega}(\rho), \quad (15)$$

where $\rho(\mu) = kR_0(\mu)$, $F_{l\Omega} = F_{l\Omega}[\rho(1)]$, and $R_0(\mu)$ determines the surface defining the shape of the nucleus according to (3). We use elliptic coordinates [see (14.1)]. If the weak dependence on ρ of the logarithmic derivatives $f_{l\Omega}$ and $g_{l\Omega}$ on the nuclear surface is neglected, the boundary condition (11) for the function leads to the equation

$$\varphi_{l\Omega}(\mu)\Phi_{l\Omega}(\mu) = \sum_{l'}{}' \beta_{ll'}^{\Omega}\gamma_{l'\Omega}(\mu)\Phi_{l'\Omega}(\mu) + i\sum_{l'}{}' \beta_{ll'}^{\Omega}\varphi_{l'\Omega}(\mu)\Phi_{l'\Omega}(\mu), \quad (16)$$

where

$$b_{ll'}^{\Omega} = -\beta_{ll'}^{\Omega}F_{l\Omega}(f_{l\Omega} + iK_{\text{eff}})/G_{l\Omega}(g_{l\Omega} + iK_{\text{eff}}). \quad (17)$$

The coefficients $\beta_{ll'}^{\Omega}$ can now be found by the same method as that used by Gol'din et al.[4] for the approximate solution of the problem of the α decay of a nonspherical nucleus. Thus we multiply (17) by $\Phi_{n\Omega}^*(\mu)/\gamma_{n\Omega}(\mu)$ and integrate over μ. At energies of the incoming particle below the height of the Coulomb barrier B, the ratio $\gamma_{l\Omega}(\mu)/\gamma_{n\Omega}(\mu)$ depends weakly on μ in the quasiclassical approximation (which we are considering here). (The functions $\gamma_{ln}(\mu)$ themselves do, of course, fluctuate strongly over the surface of the nucleus). Therefore all integrals

$$\int_{-1}^{1}\Phi_{n\Omega}^*(\mu)\frac{\gamma_{l\Omega}(\mu)}{\gamma_{n\Omega}(\mu)}\Phi_{l\Omega}(\mu)\,d\mu \quad (18)$$

are small for $n \neq l$ and can be neglected in our approximation (a special estimate shows that this involves an error of order $\beta = \Delta R/R$).

For the $\beta_{ll'}^{\Omega}$ we then obtain the following system of algebraic equations:

$$\beta_{ln}^{\Omega} = I_{ln}^{\Omega} - i\sum_{l'}\beta_{ll'}^{\Omega}I_{l'n}^{\Omega}(f_{l'\Omega} + iK_{\text{eff}}) \times F_{l'\Omega}/(g_{l'\Omega} + iK_{\text{eff}})G_{l'\Omega}, \quad (19)$$

where

$$I_{ln}^{\Omega} = \int\Phi_{n\Omega}^*(\mu)\frac{\varphi_{l\Omega}(\mu)}{\gamma_{n\Omega}(\mu)}\Phi_{l\Omega}(\mu)\,d\mu. \quad (20)$$

The terms under the sum on the right hand side of (19) are proportional to the factor $F_{l\Omega}/G_{l\Omega}$, which decreases fast with increasing l (roughly speaking, as $\exp(-l^2/\eta^{4/3})$. In a first approximation we can therefore retain only the first 2 or 3 terms in the sum. The $\beta_{ll'}^{\Omega}$ can then easily be expressed in terms of the integrals I_{ln}^{Ω}. If desired, the discarded terms can be accounted for by a method of successive approximations. In order to calculate I_{ln}^{Ω}, one must know the functions $F_{l\Omega}(\rho)$ and $G_{l\Omega}(\rho)$ near the classical turning point ρ_0. Since we consider this problem in the quasiclassical approximation, we can use the approximate solution of equation (17.1) in terms of the Airy function:[5,6]

$$F_{l\Omega}(\rho) = (t/g)^{1/4}v(-t), \quad (21)$$

$$G_{l\Omega}(\rho) = (t/g)^{1/4}u(-t), \qquad t = \left[\frac{3}{2}\int_{\rho_0}^{\rho}g^{1/2}\,d\rho\right]^{2/3}, \quad (22)$$

$$g(\rho) = 1 - \frac{2\eta\rho}{\rho^2 - c^2} - \frac{\Lambda_{l\Omega} - c^2}{\rho^2 - c^2} - \frac{c^2(\Omega^2 - 1)}{(\rho^2 - c^2)^2}, \tag{23}$$

where ρ_0 is the value of ρ for which $g(\rho) = 0$. v and u are the Airy functions tabulated by Fock.[6]

We can simplify the integral (23) by making use of the fact that we are only interested in energy values E very close to B, and therefore only in values ρ close to ρ_0. We expand the expression under the root sign in powers of $\rho_0 - \rho$ and retain only the first term. Then we have

$$F_{l\Omega}(\rho) = C^{-1/6} v(-x), \qquad G_{l\Omega}(\rho) = C^{-1/6} u(-x), \tag{24}$$

where

$$x = C^{1/3}(\rho - \rho_0), \qquad C = dg/d\rho \quad \text{for } \rho = \rho_0.$$

It is easily seen that the next term is of order $\eta\,[(B-E)/B]^{5/2}$. The above expression is therefore valid if the latter quantity is small compared to unity.

3. LIMITING VALUE OF THE AMPLITUDES $b_{ll'}^{\Omega}$ for $E < B$

We obtain a particularly simple result, if the energy of the incoming particle is so much smaller than the barrrier height, that the sum on the right hand side of equation (19) can be neglected altogether.

For this we must satisfy the condition

$$\eta\,[(B-E)/B]^{3/2} > 1, \tag{25}$$

since the "radial" functions $F_{l\Omega}(\rho)$ and $G_{l\Omega}(\rho)$ depend on the number l in essentially the same way as the corresponding Coulomb functions F_l and G_l, so that

$$\frac{F_{l\Omega}(\rho)}{G_{l\Omega}(\rho)} \leqslant \frac{F_0(\rho)}{G_0(\rho)} \approx \exp\left\{-\frac{8}{3}\eta\left(\frac{B-E}{B}\right)^{3/2}\right\}.$$

The applicability of the adiabatic approximation used by us requires that, together with (25), also

$$(B-E)/B \ll 1. \tag{26}$$

The integral I_l^{Ω} can be computed approximately by using the fact that the function $\varphi_{l\Omega}(\mu)/\gamma_{n\Omega}(\mu)$ has a sharp maximum at $|\mu| = 1$.

Introducing the new variable $\mu = \cos\theta$, we can write, for $\theta \ll 1$:

$$\varphi_{l\Omega}(\cos\theta)/\gamma_{n\Omega}(\cos\theta) \approx e^{-a\theta^2}, \tag{27}$$

where

$$a = [f_{l\Omega} - g_{n\Omega}]\, d\rho(\theta)/d(\theta^2) \quad \text{for } \theta = 0. \tag{28}$$

Since $d\rho/d(\theta^2) \approx 2\eta\beta$, the quasi-classical estimate

$$f_{l\Omega} \approx -g_{l\Omega} \approx \left(\frac{B-E}{B}\right)^{1/2},$$

leads to

$$a \approx 4\eta\beta\left(\frac{B-E}{B}\right)^{1/2}.$$

If the conditions (25) and (26) are satisfied, this quantity is practically always large in comparison with unity, and therefore only the values $|\mu|$ close to unity (i.e., θ close to zero) are essential in the integral (20).

Since $\theta \ll 1$, $\Phi_{l\Omega}(\cos\theta) \sim \theta^{\Omega}$, we have $I_{ln}^{\Omega} \sim 1/a^{2\Omega+1}$. Up to terms of order $1/a^2$ we can therefore assume

$$I_{ln}^{\Omega} = 0 \quad \text{for } \Omega \neq 0. \tag{29}$$

For the evaluation of I_{ln}^{0} one can use the representation (20.1) for the function $\Phi_{l\Omega}$, replacing $Y_{l0}(\theta, \varphi)$ by $\sqrt{(2l+1)/4\pi}\; J_0[(l+\frac{1}{2})\theta]$ and integrating over θ from zero to infinity. We then obtain the following result for the amplitudes $b_{ll'}^{\Omega}$:

$$b_{ll'}^{\Omega} = 0, \qquad \Omega \neq 0, \tag{30}$$

$$b_{ll'}^{0} = -\frac{(f_{l0} + iK_{\text{eff}})\,F_{l0}}{(g_{l'0} + iK_{\text{eff}})\,G_{l'0}}\,\frac{1}{a}\sum_{n,k} C_{ln}C_{l'k}\,[(n+1/2)(k+1/2)]^{1/2}$$

$$\times \exp\left[-\frac{(n+1/2)^2 + (k+1/2)^2}{4a}\right] I_0\left[\frac{(n+1/2)(k+1/2)}{2a}\right], \tag{31}$$

where I_0 is the Bessel function of imaginary argument.

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