SIMPLIFICATION OF THE FORMULAS FOR STATISTICAL WEIGHT CALCULATIONS

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Submitted to JETP editor October, 24, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 1303-1304 (April, 1959)

THE calculation of statistical weights has been the subject of consideration in many papers (see, for instance, references 1 and 2). The most complete results (in series form) are quoted in the review by Belen'kiĭ et al.[2] These calculations can be considerably simplified.

By definition

$$W_N = \int d\mathbf{p}_1 \ldots d\mathbf{p}_N\, \delta\left(\sum \mathbf{p}_k - \mathbf{P}_0\right) \delta\left(\sum E_k - E_0\right)$$

$$= (2\pi)^{-4} \int d^4\tau \exp\{-i\tau_\mu P_\mu^0\} \prod_k \int d\mathbf{p}_k \exp\{i(\boldsymbol{\tau}\mathbf{p}_k - \tau_0 E_k)\} \quad (1)$$

(here summation over repeated indices is to be understood, and $\tau_\mu p_\mu^0 = \boldsymbol{\tau}\cdot\mathbf{P}_0 - \tau_0 E_0$). The basic difficulty in evaluating the integral (1) is that the momentum integrals are very cumbersome, since $E_k = \sqrt{\mathbf{p}_k^2 + m_k^2}$. The difficulty is removed if we go over to the four-dimensional relativistically invariant functions

$$J_k = \int d\mathbf{p}_k \exp\{i(\boldsymbol{\tau}\mathbf{p}_k - \tau_0 E_k)\}$$

$$= (2\pi)^3 \frac{\partial}{\partial \tau_0} [\Delta(\tau_0^2 - \boldsymbol{\tau}^2) + i\Delta_1(\tau_0^2 - \boldsymbol{\tau}^2)] \quad (2)$$

(see the definition of the Δ function in reference 3). The next step usually consists in substituting for the Δ-functions their expressions in terms of Hankel functions,[2] after which, in practice, the only acceptable way to use the integral (1) is to expand it in series.

If, however, we take advantage of the well-known parametric representation of the Δ function[3]

$$\Delta = \frac{1}{2\pi^2} \int_0^\infty \cos[\alpha(\tau_0^2 - \boldsymbol{\tau}^2) + m_k^2/4\alpha]\, d\alpha,$$

$$\Delta_1 = -\frac{1}{2\pi^2} \int_0^\infty \sin[\alpha(\tau_0^2 - \boldsymbol{\tau}^2) + m_k^2/4\alpha]\, d\alpha, \quad (3)$$

we obtain for J_k

$$J_k = -8\pi i \tau_0 \int_0^\infty \alpha_k d\alpha_k \exp[-i\alpha_k(\tau_0^2 - \boldsymbol{\tau}^2) - im_k^2/4\alpha_k]. \quad (4)$$

Now the integration with respect to τ in Eq. (1) is carried out without any preliminary series expansion of the integrand, making use of the following formula:[4]

$$\int d^4\tau \exp(-i\tau P + ia\tau^2) = (\pi^2/ia^2)\exp(-iP^2/4a). \quad (5)$$

As a result we obtain

$$W_N = (2\pi)^{-4} \frac{\pi^2}{i} \left(-8\pi \frac{\partial}{\partial E_0}\right)^N \int_0^\infty \frac{\alpha_1 d\alpha_1 \ldots \alpha_N d\alpha_N}{(\alpha_1 + \ldots + \alpha_N)}$$

$$\times \exp\left\{-i\frac{\mathbf{P}_0^2 - E_0^2}{4(\alpha_1 + \ldots + \alpha_N)} - \frac{i}{4}\sum_k \frac{m_k^2}{\alpha_k}\right\}. \quad (6)$$

By a change of variables, this formula can be put into a different form:

$$W_N = \frac{1}{i\pi^2}\left(-\frac{\pi}{2}\frac{\partial}{\partial E_0}\right)^N (E_0^2 - \mathbf{P}_0^2)^{2N-2} \int_0^\infty \frac{\beta_1 d\beta_1 \ldots \beta_N d\beta_N}{(\beta_1 + \ldots + \beta_N)^2}$$

$$\times \exp\left\{\frac{i}{\beta_1 + \ldots + \beta_N} - \frac{i\nu_1^2}{\beta_1} - \ldots - \frac{i\nu_N^2}{\beta_N}\right\}, \quad (7)$$

where

$$\nu_k^2 = m_k^2/(E_0^2 - \mathbf{P}_0^2). \quad (8)$$

For $\nu_k^2 = 0$ (the ultra-relativistic case), there is no branch point in the integral (7), and a straightforward calculation leads to the well-known expression[1,5]

$$W_N^{(0)} = \left(\frac{\pi}{2}\right)^{N-1} \frac{(4N-4)!}{(3N-4)!} \frac{E_0^{3N-4}}{(2N-2)!\,(2N-1)!} \quad (9)$$

(Here P_0 has been put equal to zero, i.e., the center-of-mass system is used).

A correction to this formula, taking into account the fact that $\nu_k^2 \neq 0$, can be calculated if $\exp(-i\Sigma\nu_k^2/\beta_k)$ is expanded in a series broken off at the second term:

$$W_N^{(1)} = -\left(\frac{\pi}{2}\right)^{N-1} \frac{(4N-6)!}{(3N-6)!} \frac{E_0^{3N-4}}{(2N-3)!\,(2N-2)!} \sum_{k=1}^N (\nu_k)^2. \quad (10)$$

There are no logarithmic terms present in this case, either, due to the presence of the β_k factors in the denominator of the integrand. However, they do make their appearance in the next approximation ($\sim \nu^4$).

Thus, Eq. (6) [or (7)] appears to be a suitable starting point for approximate representations of statistical weights.

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Translated by D. C. West
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STATISTICAL ELECTRON CAPTURE MECHANISM IN BETATRONS

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Submitted to JETP editor November 21, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 1305-1306 (April, 1959)

RECENT papers have shown that the capture of injected electrons into orbits is due to the Coulomb interaction and that the capture of part of the electrons becomes possible as a result of the loss of the remaining ones. Kovrizhnykh and Lebedev succeeded by an ingenious formulation of the kinetic equations in obtaining some general result.[1] However, the mathematical difficulties of integrating the kinetic equations hide the physical picture of the process.

Physically, the considered capture mechanism, which was proposed by Matveev,[2,3] is of importance in the initial phases of the capture process, although it in itself cannot lead to capture. This is a direct consequence of Poincare's theorem on conservative systems, if the following experimental data are considered.

1) The time dependence of the magnetic field does not have an important influence on the capture. A direct experimental proof is contained in our earlier work,[4,5] where all experiments on the capture were carried out in a dc magnetic field.

2) The capture takes place also on the flat portion of the injection pulse.[6] In contrast to Logunov et al., we find[5] that the pulse fronts do not seem to be important in the injection into a dc field.

3) The captured charge increases with increasing injection pulse length and reaches its maximum value at a pulse length corresponding to several tens of revolutions. The larger the injection current the sooner saturation sets in. However, if the injection time is of the order of one revolution the captured charge is very small even for very large injection currents.[5]

From this one obtains the following picture of the process of injection into a dc magnetic field. A few revolutions after the begin of the injection there appears a stationary state in the doughnut — the number of the injected electrons almost equals the number of lost electrons. Let us take a look at the Hamiltonian for an arbitrary electron which moves in the field of all the other electrons. As long as we do not take into account the microstructure of the charge distribution, the Hamiltonian does not depend on the time. If the electron does not strike the walls it will hit the injector after a few revolutions.

The system is not conservative if one takes into account the statistical fluctuations of the charge density. As a result of the collective interaction of the electrons, their density distribution approaches statistical equilibrium.

The author[7] has investigated the equilibrium state of a toroidal electron beam employing several simplifying assumptions. The equilibrium state is determined by two parameters:

$$a = 2kT/E_0, \qquad b = P/P_0, \tag{1}$$

where k is Boltzmann's constant, kT is the mean kinetic energy of the transversal electron motion, $E_0 = qV$ (q = electron charge, V = injection voltage) is the energy of the azimuthal motion, and $P = I/V^{3/2}$, where I = beam current; $P_0 = 3.33 \times 10^{-5}$ amp/(volt)$^{3/2}$.

In the case $a \gg b$ the effective radius of the beam cross section is (in terms of the radius of the equilibrium orbit, r_0) $\rho_0 = \sqrt{a}$ and the distribution of the charge density is given by

$$\sigma/\sigma_{max} = (b/\rho_0^2) \exp\{-(\rho/\rho_0)^2\}, \tag{2}$$

where ρ is the relative distance from the beam center and $\sigma_{max} = 2\epsilon_0 V/r_0^2$ is the maximum possible charge density (in MKS units). Experiments show that the distribution of the charge density agrees well with Eq. (2) and becomes established a short time after the injection. The figure shows the experimentally-observed equal-density curves in percent of σ_{max}; the values shown in a and b were measured after 20 and 100 revolutions after the termination of the injection, respectively.

The statistical electron capture mechanism can be understood if one considers the cooling process of the beam. The loss of electrons to the walls and to the injector does not only decrease the number of the electrons in the beam but also decreases the mean transversal energy of the remaining electrons. As a result of this cooling the cross section of the beam decreases. The change of the number of electrons, ΔN, and the temperature parameter, Δa, during the time