THE BASIC COMPENSATION EQUATION IN SUPERCONDUCTIVITY THEORY WHEN THE COULOMB INTERACTION IS TAKEN INTO ACCOUNT

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Fröhlich's model is studied by the Bogolyubov method taking the Coulomb interaction into account. A partial summation of the perturbation theory series is performed by approximate second quantization in order to eliminate the infrared divergence. The basic compensation equation for dangerous diagrams and an expression for the renormalized single fermion excitation energy for the case when the Coulomb interaction is taken into account are obtained in explicit form as a result and are discussed.

1. INTRODUCTION

THE authors of recent papers on the microscopic theory of superconductivity have started from Fröhlich's model in which the Coulomb interaction between the electrons is not taken into account. At the same time it is undoubtedly impossible to assume the Coulomb repulsion to be small in real metals and the expediency of this essential simplification can only be discussed as the result of a detailed analysis of the consequences of Fröhlich's model, taking the Coulomb interaction explicitly into account. It is, however, perfectly obvious that in order to establish a criterion for the occurrence of superconductivity it is necessary to take the Coulomb interaction explicitly into account in Fröhlich's model as the principal competing factor with the electron-phonon interaction which makes the superconducting state, as is well known, the energetically more favorable. In other words, if the Coulomb effects are not taken into account, all metals described by the Fröhlich model will be superconductors which is in contradiction with the experimental facts. Bardeen, Cooper, and Schrieffer,[1] evading a mathematical consideration of this problem, assumed that a necessary condition for the occurrence of superconductivity in a given metal is the predominance of the interelectronic attraction caused by the electron-phonon interaction over the Coulomb repulsion (as an average effect in a well defined region of momentum space).

A mathematically rigorous consideration of the influence of the Coulomb interaction on superconductivity was given in the monograph by Bogolyubov, Tolmachev, and Shirkov.[2] A qualitative analysis shows that the criterion proposed in reference 1 does not correspond to reality and that the influence of the Coulomb interaction is appreciably weakened due to the collective interaction.

In the present paper we shall be engaged in a detailed investigation of the basic compensation equation, in particular of the expansion of the kernel of this integral equation in the region of the infrared divergence.

It is then essential to use a method of approximate second quantization to sum a special class of diagrams. Such a method was worked out by Bogoyubov, Tolmachev and Tyablikov and used by the authors in reference 3 to investigate the energy spectrum of a high density electron gas.

2. THE BASIC COMPENSATION EQUATION OF DANGEROUS DIAGRAMS

We start with an investigation of a dynamic system with the Hamiltonian of the Fröhlich model in the usual form, supplemented by the Coulomb interaction energy H_C:

$$H = \sum_{ks}(E(k)-\lambda)\, a^+_{\mathbf{k}s} a_{\mathbf{k}s} + \sum_{|\mathbf{q}|<q_D} \omega(q)\, b^+_{\mathbf{q}} b_{\mathbf{q}} + H_{ph} + H_c, \tag{1}$$

where

$$H_{ph} = \sum_{\substack{\mathbf{k}\mathbf{q}s \\ \mathbf{k}'-\mathbf{k}=\mathbf{q}}} (g^2\omega(q)/2\Omega)^{1/2}\,[a^+_{\mathbf{k}s} a_{\mathbf{k}'s} b^+_{\mathbf{q}} + a^+_{\mathbf{k}'s} a_{\mathbf{k}s} b_{\mathbf{q}}],$$

$$H_c = \sum_{\substack{\mathbf{p}\mathbf{p}'\mathbf{q} \\ ss'}} \frac{\nu(q)}{2\Omega}\, a^+_{p+\mathbf{q},\, s}\, a^+_{\mathbf{p}'-\mathbf{q},\, s'}\, a_{\mathbf{p}',\, s'} a_{\mathbf{p},\, s},$$

$a^+_{\mathbf{k}s}$ and $a_{\mathbf{k}s}$ are the creation and annihilation operators of an electron of momentum **k** and spin s; b_q^+ and b_q the creation and annihilation operators of a phonon with momentum $\mathbf{q}$, $\omega(q)$ and $E(k)$ the energy eigenvalues of the phonon and the electron respectively, λ the chemical potential, g the coupling constant, Ω the normalizing volume, and $\nu(q)$ the repulsive Coulomb potential in momentum representation.

One must note that the mathematical treatment of the Hamiltonian (1) is a very complicated problem. It is well known that the first convincing results for the Fröhlich model (without the term H_C) was obtained only recently in the well known paper of Bogolyubov's[4] thanks to a treatment by a new methematical method: "the principle of the compensation of dangerous diagrams." For an electron gas with a purely Coulomb interaction the situation is no less complicated and only recently has considerable success been achieved in the high-density approximation.[5]

According to the present-day formalism of perturbation theory the quantities characterizing any dynamical system can be constructed by means of the operator $R(E)$ only:

$$R(E) = H_{int} + H_{int}\frac{1}{E-H_0}H_{int} + \dots \tag{2}$$

We have, for instance, for the ground state energy and the energy of a one-fermion excitation

$$E_0 = E(\Phi_V) + \Delta E,\ E(k).$$
$$= (E(\Phi_1) - E(\Phi_V)) + \bar{E}(k), \tag{3}$$

where

$$\Delta E = \langle\Phi_V|R(E(\Phi_V))|\Phi_V\rangle_C,\quad \bar{E} = \langle\Phi_1|R(E(\Phi_1))|\Phi_1\rangle_C;$$

$E(\Phi_V)$ and $E(\Phi_1)$ are respectively the energy of the ground state Φ_V and the energy of the one-particle excitation Φ_1 when there is no interaction. The index C indicates that in the averaging only connected diagrams are taken into account.

The basic difficulty is that the equations of the usual perturbation theory (3) do not give correct results if applied to the Hamiltonian (1) for two reasons. Firstly, the electron-phonon interaction, however small, is very important near the Fermi surface and changes the structure of the energy spectrum. Furthermore, as far as the Coulomb interaction is concerned, the different terms of the perturbation theory series diverge in the infrared region. The physical crux of such a situation was analyzed carefully in Bohm and Pines' papers on plasma theory. In references 3 and 5 rigorous mathematical methods were developed to overcome this difficulty in the high density approximation which essentially consists of a partial summation of the infinite series (3). In the present paper we shall overcome both difficulties mentioned.

Following the basic idea of Bogolyubov's[4] we shall first of all perform a canonical transformation in the Hamiltonian

$$a_{\mathbf{k}\,1/2} = u_k\alpha_{k0} + v_k\alpha^+_{k1},\quad a_{-\mathbf{k}-1/2} = u_k\alpha_{k1} - v_k\alpha^+_{k0},$$
$$u_k^2 + v_k^2 = 1. \tag{4}$$

The transformed Hamiltonian can be written in the form

$$H = U + H_0 + H_{ph} + H_c + H', \tag{5}$$

where

$$U = 2\sum_{\mathbf{k}}(E(k)-\lambda)v_k^2;$$

$$H_0 = \sum_{\mathbf{k}}\tilde{\varepsilon}(k)\left[\alpha^+_{k0}\alpha_{k0} + \alpha^+_{k1}\alpha_{k1}\right] + \sum_{\mathbf{q}}\omega(q)b^+_q b_q;$$

$$H_{ph} = \sum_{\substack{\mathbf{kq}\\ \mathbf{k'-k=q}}}(g^2\omega(q)/2\Omega)^{1/2}(b^+_q + b_{-q})\left[M^+(\mathrm{k},\mathrm{k'})(\alpha^+_{k0}\alpha^+_{k1} + \alpha_{k1}\alpha_{k0}) + M^-(\mathrm{k},\mathrm{k'})(\alpha^+_{k0}\alpha_{k0} + \alpha^+_{k1}\alpha_{k1})\right];$$

$$H' = \sum_{\mathbf{k}}\left[(E(k)-\lambda)(u_k^2 - v_k^2) - \tilde{\varepsilon}(k)\right](\alpha^+_{k0}\alpha_{k0} + \alpha^+_{k1}\alpha_{k1}) + 2\sum_{\mathbf{k}}(E(k)-\lambda)u_k v_k(\alpha^+_{k0}\alpha^+_{k1} + \alpha_{k1}\alpha_{k0});$$

$$M^+(\mathrm{k},\mathrm{k'}) = u_{k'}v_k + u_k v_{k'},\quad M^-(\mathrm{k},\mathrm{k'}) = u_k u_{k'} - v_k v_{k'}.$$

The transformed H_i is a quartic expression in the Fermi operators α^+_{ks}, α_{ks} with coefficients depending on u_k and v_k.

Applying perturbation theory to the Hàmiltonian (5) we get a series which contains divergences of two kinds. All terms which contain "vacuum-two fermion" type matrix elements diverge logarithmically near the Fermi surface. To overcome this difficulty the canonical transformation (4) was developed as well as the principle of the compensation of dangerous diagrams according to which we may put these "dangerous" terms equal to zero by an appropriate choice of the u_k and v_k, namely by determining them from the equation

$$\langle \alpha_{\mathbf{k}1}\alpha_{\mathbf{k}0}R(0)\rangle = 0. \tag{6}$$

The renormalized fermion energy $\tilde{\epsilon}$ is determined from the condition that the radiative correction vanishes

$$\langle \alpha_{\mathbf{k}0}R(\tilde{\varepsilon}(k))\alpha_{\mathbf{k}0}^{+}\rangle = 0. \tag{7}$$

Equation (6) is the formal writing down of the basic compensation equation. It can, however, not be analyzed as long as $R(0)$ is in the form of an infinite series with divergent terms connected with the scattering by a Coulomb potential in the long-wavelength region. In the following we shall follow an idea of Gell-Mann and Brueckner and sum the most divergent terms of the series (6). This procedure enables us to remove the infrared divergence noted above and to evaluate accurately the main terms in the regular expansion in the dimensionless parameter $r_s = r_0/a$, where $r_0 = (4\pi N/3\Omega)^{-1/3}$ and a is the Bohr radius (r_s is small for high densities). This program can be carried out without special difficulties is the momentum $\mathbf{k}$ in Eqs. (6) and (7) corresponds to an energy $\tilde{\epsilon}(k) \sim 0$. This assumption is perfectly legitimate, although it is not completely obvious that the expansion parameter is small, since, just in the immediate neighborhood of the Fermi surface where $\tilde{\epsilon}(k) \sim 0$ the "vacuum-two fermion" type matrix elements become "dangerous." It follows therefore from the idea of the compensation of dangerous diagrams itself that we should investigate only the asymptotic behavior of Eq. (6) for $\tilde{\epsilon}(k) \sim 0$.

Moreover, Eqs. (6) and (7) can, according to Shirkov's investigation (see reference 2), be written for $\tilde{\epsilon}(k) \sim 0$ in the form

$$\langle \alpha_{\mathbf{k}1}\alpha_{\mathbf{k}0}R\rangle_c = 2\xi(k)\,u_{\mathbf{k}}v_{\mathbf{k}} - (u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2)\sum_{\mathbf{k}'} Q(\mathbf{k},\mathbf{k}')\,u_{\mathbf{k}'}v_{\mathbf{k}'} = 0, \tag{6'}$$

$$\langle \alpha_{\mathbf{k}0}R\alpha_{\mathbf{k}0}\rangle_c = \tilde{\varepsilon}(k) - (u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2)\,\xi(k) - 2u_{\mathbf{k}}v_{\mathbf{k}}\sum_{\mathbf{k}'} Q(\mathbf{k},\mathbf{k}')\,u_{\mathbf{k}'}v_{\mathbf{k}'} = 0. \tag{7'}$$

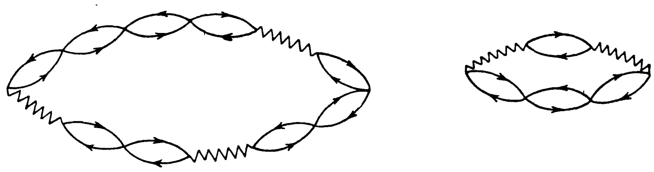
We shall here not give the complicated formal expressions for $Q(\mathbf{k},\mathbf{k}')$ and $\xi(k)$ in terms of variational derivatives with respect to Fermi amplitudes which were obtained in reference 2, since in the following we shall obtain explicit expressions for them using the simpler method of approximate second quantization.

Rigorously following a previous paper by the present authors[3] we shall write down the model Hamiltonian*

$$\begin{aligned}\tilde{H} &= \sum_{\mathbf{pq}}\tilde{\omega}(p,q)\,B_{\mathbf{p}}^{+}(\mathbf{q})\,B_{\mathbf{p}}(\mathbf{q}) + \sum_{\mathbf{q}}\omega(q)\,b_{\mathbf{q}}^{+}b_{\mathbf{q}} \\ &+ \sum_{\mathbf{pp'q}}(\nu(q)/2\Omega)\,[M^{+}(\mathbf{p},\mathbf{p}+\mathbf{q})\,M^{+}(\mathbf{p}',\mathbf{p}'-\mathbf{q}) \\ &\times(B_{\mathbf{p}}^{+}(\mathbf{q})\,B_{\mathbf{p}}^{+}(-\mathbf{q}) + B_{\mathbf{p}}(\mathbf{q})\,B_{\mathbf{p}'}(-\mathbf{q})) + 2M^{+}(\mathbf{p},\mathbf{p}+\mathbf{q})\,M^{+} \\ &\times(\mathbf{p}',\mathbf{p}'+\mathbf{q})\,B_{\mathbf{p}}^{+}(\mathbf{q})\,B_{\mathbf{p}}(\mathbf{q})] + \sum(g^2\omega_{\mathbf{q}}/2\Omega)^{1/2}M^{+}(\mathbf{p},\mathbf{p}+\mathbf{q}) \\ &\times[B_{\mathbf{p}}^{+}(\mathbf{q})\,(b_{-\mathbf{q}}^{+} + b_{\mathbf{q}}) + B_{\mathbf{p}}(\mathbf{q})\,(b_{-\mathbf{q}} + b_{\mathbf{q}}^{+})],\end{aligned}$$

$$\omega(\mathbf{p},\mathbf{q}) = \tilde{\varepsilon}(\mathbf{p}+\mathbf{q}) + \tilde{\varepsilon}(\mathbf{p}), \tag{8}$$

where $B_{\mathbf{p}}^{+}(\mathbf{q})$, $B_{\mathbf{p}}(\mathbf{q})$ are the Bose-amplitudes corresponding to the indecomposable complexes $\alpha_{\mathbf{p}+\mathbf{q},0}^{+}\alpha_{\mathbf{p}1}^{+}$. A rigorous solution of the problem with the Hamiltonian (8) is equivalent to a summation of all diagrams in which the electron-hole complex is not broken up (see the figure). The



←: electron, →: hole, wavy line: phonon

coefficients are chosen in such a way that the energy denominators and vertex parts in the model problem correspond to the corresponding elements in the summed diagrams of the exact problem. If $g^2 = 0$ and $(u_{\mathbf{k}}, v_{\mathbf{k}}) = (0, 1)$ the Hamiltonian $\tilde{H}$ goes over into the model Hamiltonian[3] for an electron gas with Coulomb interactions.

It is well known that the diagonalization of $\tilde{H}$ leads to the solution of a system of homogeneous linear equations for the functions $\varphi_{\mathbf{pq}}$ and $\alpha_{\mathbf{pq}}$, $\lambda_{\mathbf{q}}$ and $\mu_{\mathbf{q}}$:

$$\begin{aligned}&(\tilde{\omega}(p\,q) + E)\,\chi_{\mathbf{pq}} + \frac{\nu(q)}{\Omega}M^{+}(\mathbf{p},\mathbf{p}+\mathbf{q})\sum_{\mathbf{p}'}[M^{+}(\mathbf{p}',\mathbf{p}'-\mathbf{q})\,\varphi_{\mathbf{p}',-\mathbf{q}} \\ &\quad + M^{+}(\mathbf{p},\mathbf{p}'+\mathbf{q})\,\chi_{\mathbf{p}',\mathbf{q}}] + (g^2\omega_{\mathbf{q}}/2\Omega)^{1/2}M^{+}(\mathbf{p},\mathbf{p}+\mathbf{q})\,(\lambda_{-\mathbf{q}} + \mu_{\mathbf{q}}) = 0, \\ &(\tilde{\omega}(p,q) - E)\,\chi_{\mathbf{p},\mathbf{q}} + \frac{\nu(q)}{\Omega}M^{+}(\mathbf{p},\mathbf{p}+\mathbf{q})\sum_{\mathbf{p}'}[M^{+}(\mathbf{p}',\mathbf{p}'-\mathbf{q})\,\chi_{\mathbf{p}',-\mathbf{q}} \\ &\quad + M^{+}(\mathbf{p}',\mathbf{p}'+\mathbf{q})\,\varphi_{\mathbf{p}'\mathbf{q}}] + (g^2\omega_q/2\Omega)^{1/2}M^{+}(\mathbf{p},\mathbf{p}+\mathbf{q})\,(\lambda_{\mathbf{q}} + \mu_{-\mathbf{q}}) = 0, \\ &(\omega_q + E)\,\mu_{\mathbf{q}} + (g^2\omega_q/2\Omega)^{1/2}\sum_{\mathbf{p}'}(M^{+}(\mathbf{p},\mathbf{q})\,\chi_{\mathbf{p}'\mathbf{q}} + M^{+}(\mathbf{p}',-\mathbf{q})\,\varphi_{\mathbf{p}',-\mathbf{q}}) = 0, \\ &(\omega_q - E)\,\lambda_{\mathbf{q}} + (g^2\omega_q/2\Omega)^{1/2}\sum_{\mathbf{p}'}(M^{+}(\mathbf{p}',\mathbf{q})\,\varphi_{\mathbf{p}',\mathbf{q}} + M^{+}(\mathbf{p}',-\mathbf{q})\,\chi_{\mathbf{p}',-\mathbf{q}} = 0.\end{aligned} \tag{9}$$

*In Wentzel's paper[6] a model Hamiltonian was considered which differed from (8) only in that there $(u_{\mathbf{k}}, v_{\mathbf{k}})$ everywhere took the trivial values (0,1).

To determine the matrix element (7) we apply the "procedure of redefining the vacuum" developed in reference 3. We shall define two "vacuum functions" Φ_V, Φ'_V for the fermions α_{ps} and α'_{ps} respectively

$$\alpha_{ps}|\Phi_V\rangle = 0, \qquad \alpha'_{ps}|\Phi'_V\rangle = 0,$$

where

$$\alpha'_{p0} = \alpha_{p0}(1-\delta_{pk}) + \alpha^+_{p1}\delta_{pk}, \qquad \alpha'_{p1} = \alpha_{p1}(1-\delta_{pk}) - \alpha^+_{p0}\delta_{pk}.$$

The Fermi amplitudes a^+_{ps}, a_{ps} are expressed in terms of α'_{p0}, α'_{p1} by the equations

$$a_{p\,1/2} = u'_p\alpha'_{p0} + v'_p\alpha^{+\prime}_{p1}, \qquad a_{-p-1/2} = u'_p\alpha'_{p1} - v'_p\alpha^{+\prime}_{p0},$$

where

$$u_p = u'_p(1-\delta_{pk}) - v'_p\delta_{pk}, \qquad v_p = v'_p(1-\delta_{pk}) + u'_p\delta_{pk}.$$

One notes easily that Φ_V is the vacuum function of the α'_{ps} fermions and Φ'_V the state in which two fermions α_{k0} and α_{k1} are present. Then

$$\langle\alpha_{k0}R(\tilde{\varepsilon}(k))\alpha^+_{k0}\rangle$$
$$= {}^1/_2(\Delta E(\Phi'_V) - \Delta E(\Phi_V)) + \langle\alpha_{k0}H_c\alpha^+_{k0}\rangle. \qquad (10)$$

For a model Hamiltonian of the kind (8) $\Delta E(\Phi_V)$ can be evaluated exactly.[3] Thus, performing the canonical transformation (4) [or (4′)] in the exact Hamiltonian and comparing the indecomposable complexes with the Bose amplitudes $B^+_p(q)$ and $B_p(q)$ we shall obtain two model Hamiltonians $\tilde{H}$ and $\tilde{H}'$ which differ in that the functions (u_k, v_k) and (u'_k, v'_k) occur in them, respectively. The magnitude of the difference ΔE for the systems $\tilde{H}$ and $\tilde{H}'$ shall also be of interest to us.

The system (9) can easily be solved. The secular equation which determines the excited energy eigenvalues E_α is of the form

$$1 + D(q,E)f(q,E) = 0, \qquad (11)$$

where

$$D(q,E) = \Omega^{-1}\left(\nu(q) - \frac{g^2\omega_q^2}{\omega_q^2 - E^2}\right),$$

$$f(q,E) = \sum_p M^{+2}(p,p+q)\,\frac{2\omega(p,q)}{\omega^2(p,q) - E^2}.$$

For ΔE we have

$$\Delta E(\Phi_V) = -\sum E_\alpha[\chi^*_{p,q}(\alpha)\chi_{p,q}(\alpha) + \mu_q^*(\alpha)\mu_q(\alpha)], \qquad (12)$$

where $\chi_{p,q}(\alpha)$ and $\mu_q(\alpha)$ are determined from (9) and the normalization condition. After changing the sum into an integral and a number of elementary calculations (cf. reference 3)* we get

$$\Delta E(\Phi_V) =$$
$$-\sum_q \frac{1}{4\pi i}\int_\Gamma [\ln(1 + D(q,E)f(q,E)) - \frac{\nu(q)}{\Omega}f(q,E)]\,dE,$$

where the contour Γ encloses the real positive axis clockwise. $\Delta E(\Phi'_V)$ is obtained from $\Delta E(\Phi_V)$ by replacing (u_k, v_k) by (u'_k, v'_k). Subtracting $\Delta E(\Phi_V)$ from $\Delta E(\Phi'_V)$ we have

$$\tilde{\varepsilon}(k) = 2u_kv_k\sum_{k'}Q(k,k')u_{k'}v_{k'}$$
$$+ (u_k^2 - v_k^2)\left[\sum_{k'}(u_{k'}^2 - v_{k'}^2)F(k,k') + \sum_{k'}\frac{\nu(q)}{\Omega}v^2_{k'+q}\right], \qquad (13)$$

where

$$Q(k,k+q) = -\frac{1}{2\pi i}\int_\Gamma \frac{D(q,E)}{1 + D(q,E)f(q,E)}\,\frac{2\omega(k,q)}{\omega^2(k,q) - E^2},$$

$$F(k,k+q) = -\frac{1}{4\pi i} \qquad (14)$$

$$\times\int_\Gamma \frac{D(q,E)f(q,E)\nu(q)/\Omega + g^2\omega_q^2/(\omega_q^2 - E^2)}{1 + D(q,E)f(q,E)} \times \frac{2\omega(k,q)}{\omega^2(k,q) - E^2}\,dE$$

Comparing (13) with (7′) we find in our approximation the equation for the compensation of dangerous diagrams:

$$2\xi(k)u_kv_k - (u_k^2 - v_k^2)\sum_{k'}u_{k'}v_{k'}Q(k,k') = 0, \qquad (15)$$

where

$$\xi(k) = \sum_{k'}(u_{k'}^2 - v_{k'}^2)F(k,k') + \sum_{k'}\frac{\nu(q)}{\Omega}v^2_{k'+q}, \qquad (16)$$

and where $Q(k,k')$ is given by Eq. (14). Introducing $C(k)$ instead of (u_k, v_k) through the formula

$$C(k) = \sum_{k'}Q(k,k')\,u_{k'}v_{k'},$$

we can write Eqs. (13) and (15) in the more compact form:

$$C(k) = \frac{1}{2}\sum_{k'}Q(k,k')C(k')/\sqrt{C^2(k') + \xi^2(k')},$$

$$\tilde{\varepsilon}(k) = \sqrt{C^2(k) + \xi^2(k)}. \qquad (17)$$

The solution of Eq. (17) enables us to construct a microscopic theory of superconductivity and in particular a criterion for the occurrence of superconductivity in an electron-phonon system.

We shall now discuss the properties of $\xi(k)$ and $Q(k,k')$. $\xi(k)$ is by definition the energy

*When the sum is changed into an integral one must take the residue at infinity into account. We note in this connection that in Eq. (4) of reference 3 the constant term should be omitted.

of a one-fermion excitation from the ground state $[C(\mathbf{k}) = 0]$ taking corrections of all orders into account. After a few transformations we can write it in the form

$$\xi(k) = (E(k) - \lambda) + \sum_{|\mathbf{k}+\mathbf{q}|<\mathbf{k}_F} \frac{\nu(q)}{\Omega}$$

$$+ \frac{1}{4\pi}\left[\sum_{|\mathbf{k}+\mathbf{q}|>\mathbf{k}_F} - \sum_{|\mathbf{k}+\mathbf{q}|<\mathbf{k}_F}\right] \int_{-\infty}^{+\infty} \left[\frac{\nu(q) f(q, u)/\Omega}{f(q, u) + D(q, u)}\right.$$

$$\left. + \frac{\omega_q^2/\Omega(\omega_q^2 + u^2)}{1 + D(q, u) f(q, u)}\right] \frac{2\omega(\mathbf{k}, \mathbf{q})}{\omega^2(\mathbf{k}, \mathbf{q}) - u^2} du,$$

where

$$f(q, u) = \sum_p \frac{2\omega(\mathbf{p}, \mathbf{q})}{\omega^2(\mathbf{p}, \mathbf{q}) + u^2}, \qquad D_q(u) = \Omega^{-1}\left(\nu(q) - \frac{g^2\omega_q^2}{\omega_q^2 + u^2}\right),$$

which for $g^2 = 0$ agrees exactly with the result of reference 3.

To study the asymptotic behavior of $Q(k, k')$ for small q we take the main term from (14) evaluating the residue at the pole $E = \omega(\mathbf{k}, \mathbf{q})$. We have then

$$Q(\mathbf{k}, \mathbf{k}+\mathbf{q}) = Q_c(\mathbf{k}, \mathbf{k}+\mathbf{q}) + Q_{ph}(\mathbf{k}, \mathbf{k}+\mathbf{q}),$$

$$Q_c(\mathbf{k}, \mathbf{k}+\mathbf{q}) = \frac{\nu(q)}{\Omega}\left[1 + \frac{\nu(q)}{\Omega} f(q, \omega(k, q))\right]^{-1}$$

$$\approx \frac{4\pi e^2}{\Omega}\left[q^2 + k_F^2 \frac{r_s\alpha}{\pi}\left(1 - \frac{1}{2x}\ln\frac{1+x}{1-x}\right)\right]^{-1},$$

$$Q_{ph}(\mathbf{k}, \mathbf{k}+\mathbf{q})$$

$$-\frac{1}{\Omega}\frac{g^2\omega_{\mathbf{q}}^2}{\omega_q^2 - \omega^2(\mathbf{k}, \mathbf{q})}\left[1 + \frac{k_F^2}{q^2}\frac{r_s\alpha}{\pi}\left(1 - \frac{1}{2x}\ln\frac{1+x}{1-x}\right)\right]^{-1},$$

where $x = k_F/(kt + \frac{1}{2}q)$, $t = (\mathbf{p}\cdot\mathbf{q})/pq$. It is then clear that $Q_c(\mathbf{k}, \mathbf{k}')$ is essentially the screened Coulomb interaction between the electrons. For small q, $Q_{ph}(\mathbf{k}, \mathbf{k}') \sim 0$ which means the predominance of the effects of the Coulomb interaction in the long-wavelength region. These calculations completely confirm the qualitative considerations given in reference 2.

3. CONCLUSIONS

1. The influence of the Coulomb interaction on the occurrence of superconductivity has not been investigated sufficiently. Taking it into account is especially important from the point of view of establishing a criterion for the occurrence of superconductivity.

2. In Bogolyubov's new method in superconductivity theory the main difficulty in taking the Coulomb interaction into account is the infrared divergence of the basic quantities which occur in the equation of compensation of dangerous diagrams.

3. According to an idea of Gell-Mann and Brueckner and using the method of approximate second quantization we summed a special class of the most divergent diagrams giving a contribution to the main terms of an expansion in r_s, thus removing the infrared divergence and obtaining explicit expressions for the functions $\xi(\mathbf{k})$ and $Q(\mathbf{k}, \mathbf{k}')$ as the basic quantities in the compensation equation.

4. Equations (6′) and (7′) together with (14) and (15) form a closed system and a detailed investigation of it from the point of view of determining a criterion for the occurrence of superconductivity in an electron-phonon system remains a problem for further study. We must, however, note that to solve this physically very interesting problem one must clearly take the influence of the periodic field of the crystalline lattice into account in a more exact manner.

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