EXCITATION OF ROTATIONAL NUCLEAR LEVELS BY CHARGED PARTICLES

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Scattering of charged particles by nuclei with large quadrupole moments is considered in the adiabatic approximation.

THE excitation of nuclei by charged particles has been thoroughly investigated both theoretically and experimentally[1] for the case when the energy of the incident particle is considerably lower than the Coulomb energy barrier B. For these conditions the nuclear forces are not involved and the excitation is controlled completely by the electromagnetic interaction, which usually may be considered as a small perturbation. Here, however, only the E2 transitions have a significant probability of occurrence (and E1 transitions occur rarely). Because of the fact that excitation cross-sections increase rapidly with an increase of collision energy, it is of interest to examine the excitation of nuclei by particles with energy close to the Coulomb barrier level. The interpretation of such experiments is complicated by two circumstances: firstly, the nuclear interaction becomes significant in this case, and secondly, the electromagnetic interaction between the particle and the nucleus may be not small. Nuclear interactions may be most simply accounted for by considering the level group excitation, when the internal structure of the nucleus remains unchanged. The nucleus in this case can evidently be described sufficiently well by a complex potential. On the other hand, the difficulties arising from the presence of a strong electromagnetic interaction will indeed be related to the group excitation, and expecially to the excitation of rotational levels.

In the present paper we have concentrated particularly on this aspect of the problem. We examine the scattering of protons or α particles by deformed nuclei for the conditions when the noncentral portion of the electrical potential cannot be considered as a small perturbation and we confine ourselves to constructing a wave function for the scattered particle in a region outside the range of the nuclear forces. We neglect in this case the possibility of excitation of vibrational or any other (non-rotational) levels by the target nuclei. Consideration of nuclear interaction will be taken up in a later paper.

1. CONSTRUCTION OF THE WAVE FUNCTION OUTSIDE OF THE NUCLEUS

We set the origin of a fixed coordinate system at the center of gravity of the excited nucleus and set the z axis in the direction of the incident beam of particles.

We describe the orientation of the nucleus by Eulerian angles (θ_i) and the position of the incident particle by its radius vector $\mathbf{r}\,(r, \theta, \varphi)$. We assume that the shape of the nucleus is axially symmetric. If the particle energy $E \cong B$, then the excitation of the low rotational levels may be considered from the adiabatic standpoint, that is, the particle is assumed scattered by a fixed nucleus. Indeed, the cross sections for Coulomb excitation obtained by perturbation theory depend on the excitation energy ΔE through the parameter $\eta\Delta E/2E$, where[2]

$$\eta = Z_1 Z_2 e^2 / \hbar v. \tag{1}$$

The condition for which the adiabatic approximation holds consists in this case of the requirement that

$$\eta \Delta E / 2E \ll 1. \tag{2}$$

It can be seen that this condition is not connected with the application of perturbation theory and may therefore also be used for the general case. Let us note that since $2\eta = kR$ when $E = B$ (k is the wave number for the particle, and R is the nuclear radius), Eq. (2), will in our case be simultaneously the criterion for the applicability of the adiabatic approximation to nuclear scattering.[3]

Assuming for purposes of computation that $R = 1.2 \times 10^{-13} A_2^{1/2}$ cm (where A_2 is the atomic number for the target nucleus), we find that the left member of formula (1) (for $E = B$) is equal to $0.08\sqrt{A_1 A_2 / Z_1 Z_2}\,\Delta E$ (Mev) and therefore inequality (1) is fulfilled for $\Delta E \leq 1$ Mev.

The scattering process of interest to us will therefore be described by the Schrödinger equation

$$\{-(\hbar^2/2m)\nabla^2 + V_n(r, \theta_i) + V_c(r, \theta_i) - E\}\Psi(r, \theta_i) = 0, \quad (3)$$

where V_n and V_c are respectively the nuclear and Coulomb potentials. The wave function Ψ of the particle for large values of r has the form

$$\Psi \to e^{ikr + i\eta \ln(kr - \mathbf{k}r)} + \frac{f(\theta, \varphi, \theta_i)}{r} e^{ikr - i\eta \ln kr}. \quad (4)$$

The amplitude b_{if} of the nuclear transition from the initial state with total momentum I_i, of its projection M_i on the fixed axis z, and of the projection K_i on the nuclear symmetry axis in the final state I_f, M_f, $K_f = K_i = I_i$ with the simultaneous scattering of the particle at solid angle $d\Omega$ is equal to

$$b_{if} = \int \widetilde{D}^{I_f *}_{M_f K_i}(\theta_i)\, f(\theta, \varphi, \theta_i)\, \widetilde{D}^{I_i}_{M_i K_i}(\theta_i)\,(d\theta_i), \quad (5)$$

where $\widetilde{D}^{I}_{MK}(\theta_i)$ is the normalized wave function for the rotational state (I, M, K).

For the total excitation cross section of the level with momentum I_f we have

$$\sigma_{if} = \frac{1}{2I_i + 1}\int d\Omega \sum_{M_i, M_f} |b_{if}|^2. \quad (6)$$

We examine Eq. (3) outside the region of nuclear force action. In this region $V_n = 0$ and V_c may be represented in the form

$$V_c = Z_1 Z_2 e^2 / r + \sum_{n=1}^{\infty} Z_1 Z_2 e^2 r^{-(2n+1)} Q^{(2n)} P_{2n}(\cos\theta'), \quad (7)$$

where P_n is a Legendre polynomial and θ' is the polar angle of the particle in the system of coordinates fixed with respect to the nucleus.

For multipole moments $Q^{(2n)}$ of highly deformed nuclei, we may take as an estimate

$$Q^{(2n)} \sim R^{2n}\beta^n, \quad (8)$$

where $\beta = \Delta R/R$ is the deformation parameter. If we make use of (8) it is not difficult to conclude that for $E \cong B$ the n-th member of summation (7) may be considered a perturbation in Eq. (3) if

$$\eta\beta^n < 1. \quad (9)$$

We shall assume henceforth

$$\eta\beta \sim 1, \quad (10)$$

$$\beta \ll 1 \quad (11)$$

and neglect in all cases quantities of order β compared with unity. Condition (10) denotes that we may consider only collisions of nuclei with protons and α particles, because for heavy ions $\eta\beta \gg 1$ at $E \cong B$, while for $E \ll B$ the adiabatic approximation is not applicable.

If condition (10) holds, then the first (quadrupole) member of summation (7) can no longer be considered a perturbation and must therefore be exactly accounted for in the solution of Eq. (3). On the other hand, the other members of this summation, in accordance with (8) and (11), remain small. With this condition in mind we substitute for potential (7) in Eq. (3) the expression

$$V_c' = {}^1/_2 Z_1 Z_2 e^2 [1/r_1 + 1/r_2], \quad (12)$$

where r_1 and r_2 are the distances from the particle to two points located on the major axis of the nucleus, symmetrically on opposite sides of the origin of coordinates at a distance d from the origin with

$$d = \sqrt{Q^{(2)}}. \quad (13)$$

Expanding Eq. (12) in Legendre polynomials in $\cos\theta'$, it is easy to prove that the first and second members of such a series coincide with the spherically symmetric and quadrupole terms of potential V_c and therefore V_c' differs from the exact potential only by a small quantity of the order of β.

Introducing the elliptical coordinates

$$\xi = (r_1 + r_2)/2d; \quad \mu = (r_1 - r_2)/2d; \ \varphi, \quad (14)$$

we can write Eq. (3) for the region outside the nucleus in the form

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\Psi}{\partial\mu}\right\} + \frac{\partial}{\partial\xi}\left\{(\xi^2 - 1)\frac{\partial\Psi}{\partial\xi}\right\} + \left(\frac{1}{1-\mu^2} + \frac{1}{\xi^2 - 1}\right)\frac{\partial^2\Psi}{\partial\varphi^2} + \{k^2 d^2(\xi^2 - \mu^2) - 2\eta k d\xi\}\Psi = 0. \quad (15)$$

This equation may be solved by separation of the variables and its particular solutions may be written as

$$\frac{R_{l\Omega}(\rho)}{\sqrt{\xi^2 - 1}}\Phi_{l\Omega}(\mu)e^{i\Omega\varphi},$$

for which $R_{l\Omega}$ and $I_{l\Omega}$ satisfy the equations

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Phi_{l\Omega}}{d\mu}\right\} + \left\{\frac{-\Omega^2}{1-\mu^2} - c^2\mu^2 + \Lambda_{l\Omega}\right\}\Phi_{l\Omega} = 0, \quad (16)$$

$$\frac{d^2 R_{l\Omega}}{d\rho^2} + \left[1 - \frac{2\eta\rho}{\rho^2 - c^2} - \frac{\Lambda_{l\Omega} - c^2}{\rho^2 - c^2} - \frac{c^2(\Omega^2 - 1)}{(\rho^2 - c^2)^2}\right]R_{l\Omega} = 0, \quad (17)$$

where $c^2 = k^2 d^2$, $\rho = \xi c$, and $\Lambda_{l\Omega}$ are the eigenvalues of Eq. (16). If $c \to 0$, $\Lambda_{l\Omega} \to l(l+1)$, $\mu \to \cos\theta'$, and Eq. (16) is transformed into the equation for the associated Legendre polynomials.

We will consider $\Phi_{l\Omega}(\mu)$ normalized so that

$$\int_{-1}^{1} \Phi_{l'\Omega}(\mu)\,\Phi_{l\Omega}(\mu)\,d\mu = \delta_{ll'}, \tag{18}$$

$$\Phi_{l\Omega}(\mu)\,e^{i\Omega\varphi} \to Y_{l\Omega}(\theta'\varphi') \text{ for } c \to 0, \tag{19}$$

where $Y_{l\Omega}(\theta'\varphi')$ is a spherical function. It is convenient to represent the "angle" functions $\Phi_{l\Omega}$ in the form

$$\Phi_{l\Omega}(\mu)\,e^{i\Omega\varphi} = \sum_n{}' c_{ln}^{\Omega} Y_{n\Omega}(\arccos\mu, \varphi). \tag{20}$$

The prime on the summation sign indicates that n takes on values of only the same parity as l. Tables exist[4] for the eigenvalues of $\Lambda_{l\Omega}$ and for the coefficients c_{ln}^{Ω} (a more detailed solution of Eq. (16) is taken up in references 4 and 5).

The "radial" equations (17) have two linearly-independent solutions, $F_{l\Omega}(\rho)$ and $H_{l\Omega}(\rho)$ and at $\rho \to \infty$

$$F_{l\Omega}(\rho) \to \sin(\rho - l\pi/2 - \eta\ln 2\rho + \sigma_{l\Omega}), \tag{21}$$

$$H_{l\Omega}(\rho) \to \exp i\,\{\rho - l\pi/2 - \eta\ln 2\rho + \sigma_{l\Omega}\}. \tag{22}$$

The nuclear and Coulomb potentials in Eq. (3) have inherently different symmetries (if, for example, we consider the nucleus to be a uniformly charged ellipsoid and the potential is taken as the approximation (12), then the surface of the nucleus will not be an equipotential). For this reason the "momentum" l is not conserved in scattering and the particular solutions for Eq. (3) describing the existence of a particle with a definite "momentum" in the incident wave will have the following form outside the nucleus

$$\Psi_{l\Omega} = F_{l\Omega}(\rho)\,\Phi_{l\Omega}(\mu)\,e^{i\Omega\varphi} + \sum_{l'}{}' b_{ll'}^{\Omega} H_{l'\Omega}\,\Phi_{l'\Omega}e^{i\Omega\varphi}. \tag{23}$$

The amplitudes of the nuclear scattering $b_{ll'}^{\Omega}$ are determined from the condition that (23) must be finite and continuous. A solution of Eq. (3) which goes asymptotically into (4) can be represented by the linear superposition of the functions (23).

$$\Psi = \sum_{l\Omega} a_{l\Omega}(\theta_i)\,\Psi_{l\Omega}. \tag{24}$$

The coefficients $a_{l\Omega}(\theta_i)$ can be easily found if we make note of the fact that for $r \to \infty$, $\mu \to \cos\theta'$ and therefore at great distances from the nucleus the entire dependence of function $\Psi_{l\Omega}$ on angles θ_i which characterize the nuclear orientation is contained in the spherical functions $Y_{n\Omega}(\theta', \varphi')$. [Here it is convenient to use form (20) for the functions $\Phi_{l\Omega}(\mu)$.]

If we examine in functions (24) and (5) the convergent wave and equate the coefficients of similar spherical functions we obtain

$$a_{l\Omega}(\theta_i) = 4\pi i^l e^{i\sigma_{l\Omega}} \sum_n{}' c_{ln}^{\Omega} Y_{n\Omega}^{*}(\theta_i). \tag{25}$$

In deriving formula (25) we use the fact that the coefficients c_{ln}^{Ω} form a unitary matrix.

Substituting (25) and (23) into (24) we obtain for the scattering amplitude,

$$f(\theta, \varphi, \theta_i) = \frac{2\pi}{ik} \sum_{ll'\Omega pn} (\delta_{ll'} + b_{ll'}^{\Omega}) \times \exp\{i(\sigma_{l\Omega} + \sigma_{l'\Omega})\} c_{l'p}^{\Omega} c_{ln}^{\Omega} Y_{p\Omega}^{*}(\theta_i)\,Y_{n\Omega}(\theta'\varphi'). \tag{26}$$

Integration in (6) and (7) and summation over M_i and M_f in (7) are now readily carried out, and we obtain finally

$$\sigma_{if} = \eta^2\pi k^{-2} \sum_L (C_{L0I_i\,I_i}^{I_f\,I_i})^2 \{f_c^{(L)} + f_n^{(L)} + 2f_{nc}^{(L)}\}, \tag{27}$$

where $C_{\cdots}^{\cdot\cdot}$ are the Clebsch-Gordan coefficients and

$$f_c^{(L)} = \sum_{k,n} \left| \sum_{l,\Omega} c_{lk}^{\Omega} c_{ln}^{\Omega} e^{2i\sigma_{l\Omega}} C_{k\Omega n-\Omega}^{L0} \right|^2, \tag{28}$$

$$f_n^{(L)} = \sum_{k,n} \left| \sum_{ll'\Omega} c_{l'k}^{\Omega} c_{ln}^{\Omega} b_{ll'}^{\Omega} e^{i(\sigma_{l\Omega} + \sigma_{l'\Omega})} C_{k\Omega n-\Omega}^{L0} \right|^2, \tag{29}$$

$$f_{nc}^{(L)} = \sum_{k,n} \operatorname{Re}\Big\{\Big(\sum_{l\Omega} c_{lk}^{\Omega} c_{ln}^{\Omega} e^{2i\sigma_{l\Omega}} C_{k\Omega n-\Omega}^{L0}\Big) \times \Big(\sum_{ll'\Omega} c_{l'k}^{\Omega} c_{ln}^{\Omega} b_{ll'}^{\Omega *} e^{-i(\sigma_{l\Omega} + \sigma_{l'\Omega})} C_{k\Omega n-\Omega}^{L0}\Big)\Big\} \tag{30}$$

are dimensionless functions.

The first of these describes the purely Coulomb excitation of the nucleus, the second the excitation as a result of nuclear interaction, and the third the interference between these two processes. We will not compute here the nuclear amplitudes $b_{ll'}^{\Omega}$, however, it is immediately clear that, owing to the quasi-classical character of the problem, these amplitudes for $E \cong B$ decay rapidly (exponentially) with increasing l and l', and therefore the summations in (29) and (30) contain only a few terms. In (28) the convergence of the summation over l, with k and n fixed, depends inherently on the magnitude of the parameter $c = k\sqrt{Q^{(2)}}$ and improves rapidly with increasing k and n. Therefore the members of the summation over n decay as $1/n^3$ for $n > \eta$. For this reason for large values of η formula (28) is not convenient for the practical computation of $f_c^{(L)}$ and we shall obtain below another expression for this function. In order to determine the phases $\sigma_{l\Omega}$ of the "radial" wave functions we may utilize the quasi-classical approximation, which when applied to Eq. (17) yields

$$\sigma_{l\Omega} = \left(l + \frac{1}{2}\right)\frac{\pi}{2} + \frac{\eta}{2}\ln[\eta^2 + \Lambda_{l\Omega} - c^2] - \eta - \sqrt{\Lambda_{l\Omega} - c^2}\arcsin\left[1 + \frac{\eta^2}{\Lambda_{l\Omega} - c^2}\right]^{-1/2} + \int^{\infty}\left[\left(1 - \frac{2\eta\rho}{\rho^2 - c^2} - \frac{\Lambda_{l\Omega} - c^2}{\rho^2 - c^2} - \frac{c^2(\Omega^2 - 1)}{(\rho^2 - c^2)^2}\right)^{1/2} - \left(1 - \frac{2\eta}{\rho} - \frac{\Lambda_{l\Omega} - c^2}{\rho^2}\right)^{1/2}\right]d\rho. \quad (31)$$

Expression (31) reduces to an elliptic integral. However, since we already consider terms of the order of $\beta \cong c^2/\rho_0^2$ as small quantities, then, to compute the phases $\sigma_{l\Omega}$ to the same degree of accuracy, we can expand the integrand in (31) in powers of c^2/ρ^2 and use only the first two members of this series. For this case we obtain

$$\sigma_{l\Omega} = \left(l + \frac{1}{2}\right)\frac{\pi}{2} + \frac{\eta}{2}\ln[\eta^2 + \Lambda_{l\Omega} - c^2] - \eta - \sqrt{\Lambda_{l\Omega} - c^2}\arcsin\left[1 + \frac{\eta^2}{\Lambda_{l\Omega} - c^2}\right]^{-1/2} - \frac{c^2\eta}{\Lambda_{l\Omega}}\left(1 - \frac{\eta}{\sqrt{\Lambda_{l\Omega}}}\arccos\frac{\eta}{\sqrt{\eta^2 + \Lambda_{l\Omega}}}\right) - \frac{c^2(\Lambda_{l\Omega} + \Omega^2 - 1)}{4\Lambda_{l\Omega}^2} \times\left[(3\eta^2 + \Lambda_{l\Omega})\frac{1}{\sqrt{\Lambda_{l\Omega}}}\arccos\frac{\eta}{\sqrt{\eta^2 - \Lambda_{l\Omega}}} - 3\eta\right]. \quad (32)$$

2. SEMI-CLASSICAL COMPUTATION OF FUNCTION $f_c^{(L)}$

The functions $f_c^{(L)}$ actually depend on two dimensionless parameters, η and $\eta\beta$. In view of the fact that in many practically important cases $\eta \gg 1$, it would be interesting to obtain a limiting value of this function for $\eta \to \infty$ and for a set value of $\eta\beta$. It is difficult to make this transition to the limit in formula (28) because of the complicated character of the relation for the coefficients $c_{\ell n}^{\Omega}$ and the characteristic values $\Lambda_{l\Omega}$ of Eq. (16) as a function of the parameters of this equation. However, from the usual theory of Coulomb excitation[2] it is known that the limit for quantum-mechanical expressions for excitation cross sections at $\eta \to \infty$ coincides with the expressions obtained on the assumption of incident particles moving in classical trajectories. There is reason to believe that this coincidence is not connected with the use of perturbation theory and that, therefore, we can obtain by analogous means the limiting values of function $f_c^{(L)}$, (28), as well.

Let us, as before, consider conditions (2), (10), and (11) satisfied, and let us look into the collision of a charged particle moving in a classical trajectory with an even-even nucleus which is in the ground state $(I_i = 0)$. The interaction of a particle with the nucleus is described by the potential (7). In view of condition (10) we can neglect the effect of the non-central portion of the Coulomb potential on the shape of the particle trajectory (this signifies neglecting corrections of the order of β), while in view of (11) we can neglect all the terms of summation (8) with $n \geq 2$. The adiabatic approximation used above is equivalent, when using this type of approach, to neglecting the kinetic energy of rotation of the nuclear target (because the adiabatic approximation signifies formally that the moment of inertia of the nucleus is considered infinite).

The motion of the nucleus is described therefore by the Schrodinger equation

$$i\hbar\partial\Psi(\theta_i)/\partial t = Z_1Z_2e^2r^{-3}(t)\,Q^{(2)}P_2[\cos\theta'(t)]\,\Psi(\theta_i) \quad (33)$$

with initial conditions $\Psi = 1/\sqrt{4\pi}$ for $t = -\infty$. Here $r(t)$ and $\theta'(t)$ are specified time functions, determined by the classical laws of motion of a particle in a trajectory. These functions depend on the particle scattering angle as a parameter (or on the eccentricity of its hyperbolic orbit). The probability of exciting a level with momentum L by a particle scattered through an angle θ is equal to

$$w_L(\theta) = \sum_{M_f}\left|a_{LM_f}(\theta)\right|^2, \quad (34)$$

where

$$a_{LM_f} = \frac{1}{\sqrt{4\pi}}\int Y^*_{LM_f}(\theta'\varphi') \times \exp\left\{-i\frac{Z_1Z_2e^2Q^{(2)}}{\hbar}\int_{-\infty}^{\infty}\frac{P_2[\cos\theta'(t)]}{r^3(t)}dt\right\}d\Omega. \quad (35)$$

In order to compute a_{LM_f} it is convenient to use a system of coordinates with an xy plane that coincides with the plane of particle motion, and an axis x directed along the axis of symmetry for the hyperbolic orbit of the incident particle. The integral in t in (35) is then expressed through the orbital integrals I_{20} and I_{22} known from the theory of Coulomb excitation (their explicit form is given in reference 1). To obtain the total excitation cross section corresponding to the level with momentum L we must multiply (34) by the cross section for Rutherford scattering and integrate it over θ.

Writing down this cross-section in the form

$$\sigma_L = \pi\eta_i^2k^{-2}f_c^{(L)}(x), \quad x = \eta_iQ^{(2)}/2\,(Z_1Z_2e^2/2E)^2, \quad (36)$$

we obtain

$$f_c^{(L)}(x) = \frac{1}{\pi}\int_{-1}^{1}\sum_m\left|\int Y_{Lm}(\theta,\varphi)\right.$$

$$\times\exp\left\{ixz^2\sin^2\theta\left[\frac{3}{1-z^2}\left(1-\frac{z\arccos z}{\sqrt{1-z^2}}\right)-\cos 2\varphi\right]\right\}$$

$$\left.\times\sin\theta d\theta d\varphi\right|^2\frac{dz}{z^3}, \qquad (37)$$

where $z = \sin(\theta/2)$. Formula (36) refers to the excitation of the even-even nucleus ($I_i = 0$); in the general case the cross section for the transition $I_i \rightarrow I_f$, as can be easily shown, will be

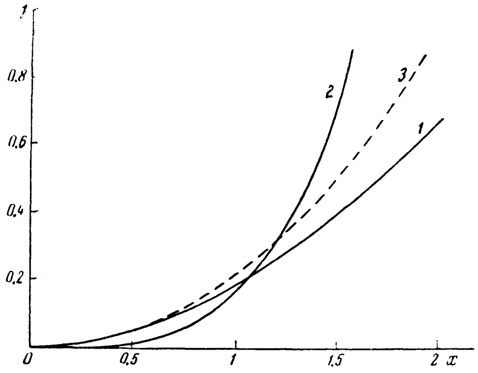
$$\sigma_{if} = \frac{\pi\eta^2}{k^2}\sum_L (C^{I_f I_i}_{L0 I_i I_i})^2 f_c^{(L)}(x). \qquad (38)$$

In accordance with what has been said above we can assume that Eq. (38) is the first term of the series expansion of the exact expression (28) in powers of $1/\eta$ [moreover, it differs from (28) by a quantity of the order of β because it is derived with the use of a potential different from (12)]; it can be used in (27). The figure shows a plot of the functions $f_c^{(2)}$ and $f_c^{(4)}$. For $x \lesssim 1$ these functions can be conveniently represented as series in powers of x; for the functions $f_c^{(2)}$ and $f_c^{(4)}$ this series has the form

$$f_c^{(2)}(x) = 0.2279x^2 - 0.015x^4 - 0.0056x^6 + \ldots$$

$$f_c^{(4)}(x) = 0.00453x^4 - 0.000407x^6 + \ldots$$

It is worth noting that functions $f_c^{(L)}$ are the even functions, owing to the use of the adiabatic approximation.



plots of functions $1 - f_c^{(2)}(x)$; $2 - f_c^{(4)}(x)\times 50$; $3 - f_c^{(2)}(x)$ by perturbation theory

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