NEW DISPERSION RELATIONS IN QUANTUM FIELD THEORY

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Submitted to JETP editor June 17, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 1088-1092 (April, 1959)

Some new dispersion relations are obtained between the modulus and the phase shift of the forward scattering amplitude. In contrast to the usual dispersion relations between the real and imaginary parts of the forward scattering amplitude, the present relations do not depend on the detailed behavior (degree of increase or decrease) of the forward scattering amplitude at infinite energy. In connection with the deduced dispersion equations the question of possible existence of zeros in the forward scattering amplitude in its region of analyticity is investigated.

1. In order to make clear the main ideas in the derivation of the new dispersion relations, we shall study a model. With appropriate technical improvements this model could be applied to realistic theories.

Let f(E) be the forward scattering amplitude defined by the Fourier integral*

$$f(E) = \int_{-\infty}^{\infty} \tilde{F}(t)\, e^{iEt}\, dt \equiv \psi(E)\, e^{i\varphi(E)} \qquad (1)$$

of a causal function $\tilde{F}(t)$, i.e., a semi-finite function:

$$\tilde{F}(t) = \begin{cases} F(t), & t > t_0 \\ 0 & t < t_0 \end{cases} \qquad t_0 \leqslant 0. \qquad (2)$$

In the case when the limit of semi-finiteness $t_0 = 0$ we say that $\tilde{F}(t)$ is microcausal, and when $t_0 < 0$ we say that $\tilde{F}(t)$ is macrocausal.[1] In the derivation of the usual dispersion relations one makes use of the condition of microcausality.[1]

The function f(E) satisfies the following symmetry condition:†

$$f(E) = f^*(-E) \qquad (3)$$

and from the "optical" theorem we have[1]

$$\operatorname{Im} f(E) = \frac{k}{4\pi}\, \sigma(E), \quad E \in [\mu, \infty), \qquad (4)$$

Here $\sigma(E)$ is the total scattering cross section and $k^2 = E^2 - \mu^2$, where μ is the rest mass of the particle.

As is well known, relations of the type (1), (3), or (4) may be derived rigorously starting from only the most fundamental physical principles (in particular, the causality principle) and, most important, without any specific assumptions about the (unknown) form of the interaction operator. The function $\tilde{F}(t)$, of course, does depend on the specific structure of the interaction operator.

The derivation of the usual dispersion relations is based on the analyticity of F(E) in the upper half plane Im E > 0, which follows from the semi-finiteness of $\tilde{F}(t)$. However, in the derivation of the usual dispersion relations between Re f(E) and Im f(E), it is necessary to make additional assumptions about the behavior of f(E) as $|E| \to \infty$.[1,2-9] It is important to note that the behavior of f(E) as $|E| \to \infty$ does not, in general, follow from the fundamental physical principles used to obtain the basic relation (1). When studying the behavior of f(E) as $|E| \to \infty$ one must start from a specific form for the interaction operator.[10,11] This is most unsatisfactory since it is believed that it is exactly at these infinitely high energies that present day theories are inconsistent (and therefore the interaction operators known at present are inconsistent also).

Therefore the usual dispersion relations cannot be used directly to verify the fundamental physical principles experimentally; the failure of the usual dispersion relations* may also be due to an incorrect assumption about the behavior of f(E) as $|E| \to \infty$.

2. For those functions f(E) which are in the Lebesgue class L^2† (i.e., are square-integrable) one obtains from the causality principle (micro- as well as macro-causality) not only analyticity

*The Fourier integral (1) is defined, generally speaking, for the class of generalized functions.[2]

†For the sake of simplicity we consider the case when the particles are described by a neutral field.

*Dispersion relations are a necessary but not sufficient (!) condition for the validity of the fundamental physical principles.

†This is the class of functions f(E) considered by Oehme.[6]

of f(E) in the upper half plane $\mathrm{Im}\, E > 0$ but also the criterion of physical realizability in quantum field theory:[12]

$$\int_{-\infty}^{\infty} \frac{|\log \psi(E)|}{1+E^2} dE < \infty. \tag{5}$$

From Eq. (5) follow, in particular, definite restrictions on the behavior of f(E) as $|E| \to \infty$, namely

$$|f(E)| = \psi(E) \geqslant A \exp\{-\gamma |E|^q\}, \tag{6}$$

where $A > 0$, $\gamma > 0$, $q < 1$, i.e., f(E) cannot decrease too rapidly.

On the other hand f(E) cannot increase too rapidly as $|E| \to \infty$ since it follows in any case from the analyticity of f(E) that

$$|f(E)| = \psi(E) \leqslant A_1 \exp\{\gamma_1 |E|^{q_1}\}, \tag{7}$$

where $A_1 > 0$, $\gamma_1 > 0$, $q_1 < 1$. Furthermore, it can be shown[1] that f(E) can increase or decrease as $|E| \to \infty$ only as a polynomial in E.

3. On the basis of the analyticity of the forward scattering amplitude f(E) in the upper half plane $\mathrm{Im}\, E > 0$ and on the basis of the criterion of physical realizability (5) and (6), and the condition (7) we obtain by methods analogous to those used in quantum decay theory[13-15] the desired dispersion relations between $\log \psi(E)$ and $\varphi(E)$, which do not depend on the degree of increase or decrease of f(E) as $|E| \to \infty$:

$$\begin{aligned}\varphi(E) = &-\frac{(E-E_0)^m}{\pi} \mathrm{P}\int_0^{\infty} \frac{\log\psi(E')\left[(E'+E)(E'+E_0)^m + (-1)^{m+1}(E'-E)(E'-E_0)^m\right]}{(E'^2-E^2)(E'^2-E_0^2)^m} dE' \\ &+ \varphi(E_0) + \frac{(E-E_0)}{1!}\frac{d\varphi(E_0)}{dE_0} + \cdots + \frac{(E-E_0)^{m-1}}{(m-1)!}\frac{d^{m-1}\varphi(E_0)}{dE_0^{m-1}} - \sum_k \tan^{-1}\frac{2(E-E_k^{(1)})E_k^{(2)}}{(E-E_k^{(1)})^2 - E_k^{(2)2}} \\ &+ \sum_k \left\{\tan^{-1}\frac{2(E_0-E_k^{(1)})E_k^{(2)}}{(E_0-E_k^{(1)})^2-(E_k^{(2)})^2} + \cdots + \frac{(E-E_0)^{m-1}}{(m-1)!}\frac{d^{m-1}}{dE_0^{m-1}}\tan^{-1}\frac{2(E_0-E_k^{(1)})E_k^{(2)}}{(E_0-E_k^{(1)})^2-E_k^{(2)2}}\right\}; \quad m \geqslant 2.\end{aligned} \tag{8}$$

If f(E) increases or decreases as a polynomial[1] as $|E| \to \infty$ then it can be shown, making use of the symmetry condition (3), that the following much simpler dispersion relation is valid:

$$\begin{aligned}\varphi(E) = &-\frac{2E}{\pi}\mathrm{P}\int_0^{\infty}\frac{\log\psi(E') - \log\psi(E)}{E'^2-E^2}dE' \\ &-\sum_k \tan^{-1}\frac{2(E-E_k^{(1)})E_k^{(2)}}{(E-E_k^{(1)})^2-E_k^{(2)2}}.\end{aligned} \tag{9}$$

In Eqs. (8) and (9) the summation over k is a summation over the possible zeros of the function f(E) in the upper half plane $E_k = E_k^{(1)} + iE_k^{(2)}$; $E_k^{(2)} > 0$; and $\psi(E)$ and $\varphi(E)$ are the differential forward scattering cross section and the phase shift of the forward scattering amplitude respectively. They are given in terms of experimentally measureable quantities by

$$\begin{aligned}\psi(E) &= ([\mathrm{Re}\, f(E)]^2 + [\mathrm{Im}\, f(E)]^2)^{1/2}, \\ \varphi(E) &= \tan^{-1}[\mathrm{Im}\, f(E)/\mathrm{Re}\, f(E)], \\ \varphi(E) &= \tan^{-1}\frac{\mathrm{Im}\, f(E)}{([\psi(E)]^2 - [\mathrm{Im}\, f(E)]^2)^{1/2}}.\end{aligned} \tag{10}$$

The symbol P indicates that the integrals are principal value integrals at the points $E' = E$ as well as $E' = E_0$.

The dispersion relations (8) and (9), in contrast to the usual ones,[1,2-8] are valid for any degree of increase or decrease of the forward scattering amplitude f(E) as $|E| \to \infty$. The dispersion relations (9), as opposed to the relations (8), do not require the knowledge of the derivatives of the phase shift, which constitutes a distinct advantage from the point of view of accuracy of the experimental data.

4. The main problem arising in the application of the new dispersion relations (8), (9) is in the determination of the possible zeros of the forward scattering amplitude f(E) in the upper half plane $\mathrm{Im}\, E > 0$. The complete solution of this problem differs for particles with vanishing ($\mu = 0$) or with finite ($0 < \mu < \infty$) rest mass.

It can be shown, using criterion (5) and condition (7), that f(E) is a function of completely regular increase and in class A in the upper half plane $\mathrm{Im}\, E > 0$.[16] Consequently one has, from the theorem by A. Pflyuger,[16] convergence of the series

$$\sum_k \left|\mathrm{Im}\frac{1}{E_k}\right| < \infty \tag{11}$$

(here E_k are the zeros of the analytic function f(E) in the upper half plane) and the existence of the density

$$\Delta_f = \lim_{|E|\to\infty} n_f(|E|)/|E|, \tag{12}$$

where $n_f(|E|)$ is the number of zeros of f(E) in a semicircle of radius $|E|$. The convergence of (12) means that "almost all" possible zeros of the forward scattering amplitude f(E) lie in the neighborhood of the real axis. It is seen from Eqs. (8) and (9) that the zeros that lie close to the real axis $E_k^{(2)} \rightarrow 0$ make a vanishingly small contribution to the dispersion relations.

5. The results obtained above concerning the possible zeros of the forward scattering amplitude $f(E)$ in the upper half plane $\text{Im } E > 0$ are based on the same fundamental physical principles that are used in the proof of dispersion relations.[1]

These results can be made considerably more precise if use is made of the additional natural physical assumption that the total cross section $\sigma(E)$ differs from zero for all finite $E \in [\mu, \infty]$,

$$\sigma(E) \neq 0. \tag{13}$$

Using Eq. (4) we obtain from the additional condition (13) for particles with zero rest mass $\mu = 0$

$$\text{Im } f(E) \neq 0, \quad E \in [0, \infty), \tag{14}$$

and for particles with finite rest mass $\mu > 0$

$$\text{Im } f(E) \neq 0, \quad E \in [\mu, \infty). \tag{15}$$

On the basic of the fundamental theorem on the number of zeros of an analytic function[17] it is not difficult to see that the existence of zeros in $f(E)$ in the upper half plane $\text{Im } E > 0$ corresponds to the vanishing of $\text{Im } f(E)$ in isolated points E_i and, further, that $\text{Re } f(E)$ should alternate in sign at neighboring points E_i.

Hence, using Eq. (14), we obtain the following final result for particles with zero rest mass $\mu = 0$. For such particles the forward scattering amplitude $f(E)$ either does not vanish at all in the upper half plane $\text{Im } E > 0$ if $\text{Re } f(0)$ and $\text{Re } f(\infty)$ have the same sign, or has one simple zero $E_1 = iE_1^{(2)}$ if $\text{Re } f(0)$ and $\text{Re } f(\infty)$ have opposite signs.

On the other hand for particles with finite rest mass $0 < \mu < \infty$ it can be shown, using the fundamental theorem[17] and the condition (15), that the forward scattering amplitude $f(E)$ has only a finite number of zeros in the upper half plane $\text{Im } E > 0$. To obtain more precise results one generally speaking needs to know the behavior of $f(E)$ in the nonphysical* region $E \in (-\mu, \mu)$. At that it is only necessary to know those values of $|E_i| < \mu$ for which $\text{Im } f(E)$ vanishes and to know the sign of $\text{Re } f(E)$ at these points E_i.

*For the π-meson-nucleon scattering, for example, the behavior of f(E) in the nonphysical region is known.

In connection with the dispersion relations derived above the following mathematical problem is of interest:[15] to find the necessary and sufficient criteria which must be satisfied by the analytic function — the forward scattering amplitude $f(E)$ — for $\infty > |E| \geq \mu$ in order that it have no zeros in the upper half plane $\text{Im } E > 0$.

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Translated by A. M. Bincer
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