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ON THE THEORY OF NUCLEAR MAGNETIC MOMENTS

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The strong spin-orbit coupling gives rise to an additional interaction between the nucleons in the nucleus and the electromagnetic field. The effect of this additional interaction on the nuclear magnetic moments is considered for spherical and nonspherical nuclei. It is shown, in particular, that about half of the anomalous deviation of the magnetic moment of Bi^{209} from the Schmidt line may be due to this effect.

1. INTRODUCTION

In the shell theory of the nucleus the spin-orbit coupling plays an important role. The corresponding operator can be written in the form

$$\mathcal{H}_{ls}' = -(\lambda\hbar/m^2c^2)\,\mathbf{s}\,[\nabla U \times \mathbf{p}], \qquad (1)$$

where λ is the phenomenological spin-orbit coupling constant, $\mathbf{s}$ is the spin operator for the nucleon, $U(\mathbf{r})$ is the self-consistent nuclear potential, and $\mathbf{p}$ is the momentum operator for the nucleon.

If the nucleus is located in an electromagnetic field, general principles prescribe that the momentum of the nucleon, $\mathbf{p}$, be replaced in the Hamiltonian by the generalized momentum, $\mathbf{p} - e\mathbf{a}/c$. In addition to the usual terms, the Hamiltonian will then contain another term connected with the interaction (1):

$$\mathcal{H}' = (\lambda e\hbar/m^2c^3)\,\mathbf{s}\,[\nabla U \times \mathbf{a}], \qquad (2)$$

where $\mathbf{a}$ is the vector potential of the electromagnetic field.

It should be noted that the interaction (2) may play an essential role in the case of a proton. For a neutron, the effective charge due to the separation of the motion of the center of charge is equal to Ze/A for electric dipole transitions. In general, the effective charge is equal to Ze/A^l for electric multipoles, and to Ze/A^{l+1} for magnetic multipoles, where l is the order of the multipole. In the case of a neutron the interaction (2), therefore, does not give a significant contribution to the magnetic moment and the magnetic dipole transitions, if only the effective charge of the abovementioned origin is considered.

Mayer and Jensen[1] first pointed out the necessity of including the spin-orbit effect in the discussion of the electromagnetic properties of nuclei. However, they did not use expression (2), and their estimates are very crude. Grechukhin[2] investigated the effect of the spin-orbit coupling on the forbidden (in the shell model) magnetic transitions of the type $d_{3/2} \to s_{1/2}$.

In the present paper we consider the effect of the interaction (2) on the magnetic moments of spherical and nonspherical nuclei with an odd number of protons. The interaction (2) is, of course, not the only reason for the deviation of the observed magnetic moments from the Schmidt lines. Nevertheless, an estimate of the magnitude of the interaction (2) is undoubtedly of interest, since it allows to isolate the role of the other effects. Furthermore, this interaction may in some cases give the main contribution to deviation of the magnetic moments from the Schmidt lines, and to the probability of the forbidden M1 transitions in nuclei with an extra proton. In the case of nuclei with an extra neutron it becoms meaningful to include the interaction (2), if there exists an effective charge owing to the interaction of the neutron with the nuclear core. The possible existence of such a charge was recently considered in the Brueckner model.[3]

For the calculation of the magnetic moments it is necessary to make use of the following relation for a constant uniform magnetic field:

$$\mathbf{a} = {}^1/_2\,[\mathbf{H} \times \mathbf{r}].$$

Hence

$$\hat{\mathcal{H}}' = \frac{\lambda e\hbar}{2m^2c^3}\,\mathbf{s}\,\nabla U \times [\mathbf{H} \times \mathbf{r}] = \frac{\lambda e\hbar}{2m^2c^3}\,\mathbf{s}\,[\mathbf{H}\,(\nabla U \cdot \mathbf{r}) - \mathbf{r}\,(\nabla U \cdot \mathbf{H})]. \qquad (3)$$

Let $H_z = H$, $H_{x,y} = 0$. We then obtain the following expression for the operator of the magnetic moment connected with this interaction:

$$\hat{\mu}_z = (\lambda e\hbar\,/\,2m^2c^3)\,[(\mathbf{s}\cdot\mathbf{r})\,\nabla_z U - s_z\,(\nabla U \cdot \mathbf{r})]. \qquad (4)$$

To calculate the radiative transition probabilities we must know the expression for the current operator corresponding to the interaction (2). This operator obviously has the form

$$\hat{\mathbf{j}} = -c\delta\hat{\mathscr{H}}'/\delta\mathbf{a} = -(\lambda e\hbar/m^2c^2)[\mathbf{s}\times\nabla U]. \qquad (5)$$

With the help of this operator we can calculate the radiative transition probabilities by well-known formulas.[4]

2. MAGNETIC MOMENTS OF SPHERICAL NUCLEI

The magnetic moment of a spherical nucleus can be written in the form of the following matrix elements:*

$$\mu = \langle J, J|\hat{\mu}_z|J, J\rangle. \qquad (6)$$

It follows that

$$\mu = \frac{\lambda e\hbar}{2m^2c^3}\left\langle J\left|r\frac{\partial U}{\partial r}\right|J\right\rangle\langle J, J|(\mathbf{s}\cdot\mathbf{n})n_z - s_z|J, J\rangle, \qquad (7)$$

where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{r}$.

First of all we calculate the angular matrix element in formula (7):

$$(\mathbf{s}\cdot\mathbf{n})n_z = (\mathbf{J}\cdot\mathbf{n})n_z = [J_zn_z + {}^1/_2(J_x - iJ_y)(n_x + in_y)$$

$$+ {}^1/_2(J_x + iJ_y)(n_x - in_y)]n_z,$$

$$\langle J, J|J_zn_z^2|J, J\rangle = \sqrt{4\pi/9}\,J[(2/\sqrt{5})\langle Y_{2,0}\rangle$$

$$+\langle Y_{0,0}\rangle] = J/2(J+1). \qquad (8)$$

Obviously,

$$\langle J, J|(J_x - iJ_y)(n_x + in_y)n_z|J, J\rangle = 0.$$

The matrix element of the last term in formula (8) is also computed without difficulty. We obtain

$$\langle J, J|(J_x + iJ_y)(n \;- in_y)n_z|J, J\rangle$$

$$= \langle J, J|(J_x + iJ_y)|J, J-1\rangle\langle J, J-1|(n_x - in_y)n_z|J, J\rangle$$

$$= (8\pi/3)\sqrt{3/20\pi}\sqrt{J}\langle J, J-1|Y_{2,-1}|J, J\rangle$$

$$= -(2J-1)/2(J+1).$$

As a result we obtain the following expression for the additional magnetic moment connected with the interaction (2):

$$\mu = \mp\frac{J+{}^1/_2}{2(J+1)}\frac{e\hbar}{2mc}\frac{\lambda}{mc^2}\left\langle J\left|r\frac{\partial U}{\partial r}\right|J\right\rangle, \qquad (9)$$

where the minus sign refers to $J = l + \frac{1}{2}$, and the plus sign to $J = l - \frac{1}{2}$. The sign of this additional moment corresponds to the sign of the deviation of the nuclear magnetic moments from the Schmidt lines.

Let us now estimate the magnitude of the additional magnetic moment (9). Observing that $\partial U/\partial r$ is different from zero only near the nuclear surface, we obtain

$$\frac{\lambda}{mc^2}\left\langle J\left|\frac{1}{r}\frac{\partial U}{\partial r}r^2\right|J\right\rangle \approx \frac{\lambda}{mc^2}R_0^2\left\langle J\left|\frac{1}{r}\frac{\partial U}{\partial r}\right|J\right\rangle,$$

where R_0 is the effective radius of the nucleus. This matrix element enters into the expression for the energy of the spin-orbit coupling

$$E_{sl} = -\frac{\lambda\hbar^2}{m^2c^2}\left\langle J\left|\frac{1}{r}\frac{\partial U}{\partial r}\right|J\right\rangle(\mathbf{s}\cdot\mathbf{l})$$

and can be obtained from the energy of the spin-orbit splitting

$$\frac{\lambda}{m^2c^2}\left\langle J\left|\frac{1}{r}\frac{\partial U}{\partial r}\right|J\right\rangle = \frac{2\Delta E_{sl}}{\hbar^2(2l+1)}. \qquad (10)$$

The diffuseness of the nuclear boundary is supposedly the same for all nuclei. The function $f(r) = \partial U/\partial r$ will then also be practically the same for all spherical nuclei. This conclusion is confirmed by the experiments on the scattering of fast electrons from nuclei, as well as by the calculations in the framework of the optical model with diffuse boundary.[5,6] The region in which this function is noticeably different from zero is of order 1.5 to 2×10^{-13} cm. The wave functions of the states without radial nodes and with $l \neq 0$ differ significantly from zero in the neighborhood of the nuclear surface in approximately the same region where $f(r)$ is different from zero. It should therefore be expected that for these states

$$\frac{\lambda\hbar^2}{m^2c^2}\left\langle J\left|\frac{1}{r}\frac{\partial U}{\partial r}\right|J\right\rangle \approx CA^{-1/3}, \qquad (11)$$

where C is an empirical constant, which can be determined, for example, from the splitting of the levels $d_{5/2} - d_{3/2}$ in O^{17} (reference 7). In the case of one-nucleon states corresponding to wave functions with radial nodes, one should expect that the abovementioned matrix element depends more strongly on A than as $A^{-1/3}$.

These qualitative considerations are confirmed in a wide range of atomic weights by shell-model calculations with a potential with diffuse boundary. Such calculations were recently carried out by Nemirovskiĭ.[6]

It should also be noted that Nilsson[8] assumed in his calculations that the matrix element depends on the atomic weight as in formula (11). Using (11) with the constant C determined in the indicated fashion, and taking $R_0 = 1.23\times10^{-13}A^{1/3}$ (reference 6) as the value of the effective radius of the charge, we obtain the following estimate:

$$\mu \approx \mp A^{1/3}\,0.11\,\frac{2J+1}{2(J+1)}\frac{e\hbar}{2mc}. \qquad (12)$$

*In formulas (6) and (7), as in the following, the second index in the matrix elements denotes the projection of the moment J.

In the case of Bi^{209}, for example, this estimate gives the value $\mu \approx +0.6\, e\hbar/2mc$. This is slightly less than half the anomalous magnetic moment of this nucleus.

3. MAGNETIC MOMENTS OF NONSPHERICAL NUCLEI

In the strong coupling approximation the wave function of the nonspherical nucleus can be written as a product of the wave function for the motion of the nucleon relative to the nuclear axis and the wave function for the collective motion:[9]

$$\psi^{J}_{\Omega, K} = \chi_{\Omega} D^{J}_{M, K}. \tag{13}$$

In this case the magnetic moment is conveniently calculated with the help of the relation

$$\mu = \langle \hat{\mu} \mathbf{J} \rangle / (J+1), \tag{14}$$

where

$$\hat{\mu} = \frac{\lambda}{mc^2} \frac{e\hbar}{2mc} [(\mathbf{s} \cdot \mathbf{r}) \nabla U(r) - \mathbf{s} (\nabla U \cdot \mathbf{r})].$$

If the deformation of the nucleus is small ($\Delta R/R \ll 1$), formula (14) may be written in the form

$$\hat{\mu} = \frac{\lambda}{mc^2} \frac{e\hbar}{2mc} \frac{\langle r (\partial U / \partial r) [(\mathbf{s} \cdot \mathbf{n})(\mathbf{n} \cdot \mathbf{J}) - (\mathbf{s} \cdot \mathbf{J})] \rangle}{J+1}, \tag{15}$$

For the matrix element we have

$$\begin{aligned} \langle \Omega, K | (\mathbf{s} \cdot \mathbf{n})(\mathbf{n} \cdot \mathbf{J}) | \Omega, K \rangle &= \langle \Omega, K | (\mathbf{j} \cdot \mathbf{n})(\mathbf{n} \cdot \mathbf{J}) | \Omega, K \rangle \\ &= \langle \Omega, K | (\mathbf{j} \cdot \mathbf{n}) | \Omega' K \rangle \langle \Omega' K | (\mathbf{n} \cdot \mathbf{J}) | \Omega K \rangle \\ &= K \langle \Omega | (\mathbf{j} \cdot \mathbf{n}) n_{z'} | \Omega \rangle, \end{aligned} \tag{16}$$

Ω, K, $n_{z'}$ are the projections of **j**, **J**, **n** on the z′ axis of the nucleus, respectively.

With the help of (16) we obtain the additional magnetic moment of nuclei with spin other than $\frac{1}{2}$ in the form

$$\mu = K \mu_{\text{int}} / (J+1),$$

where

$$\mu_{\text{int}} = \left\langle \Omega \left| r \frac{\partial U}{\partial r} [(\mathbf{j} \cdot \mathbf{n}) n_{z'} - s_{z'}] \right| \Omega \right\rangle \frac{\lambda}{mc^2} \frac{e\hbar}{2mc} \tag{17}$$

is the additional magnetic moment of the nucleon in the coordinate system defined by the nuclear axes. The order of magnitude of μ_{int} is, of course, the same as for spherical nuclei.

The formula of Bohr and Mottelson[10] for the magnetic moments,

$$\mu = g_{\Omega} J^2 / (J+1) + g_R / (J+1), \tag{18}$$

therefore remains valid for nuclei with $J = K = \Omega \geq 3/2$; g_Ω and g_R are the gyromagnetic ratios for the one-nucleon and collective motions. However, besides the usual terms, g_Ω must also contain the gyromagnetic ratio arising from the additional magnetic moment (17).

Let us now consider the magnetic moments of nuclei with spin $\frac{1}{2}$. We shall assume that the case of coupling b according to Hund[11] applies to these nuclei. The one-nucleon states in these nuclei can then be classified as Σ^{+}_{g} and Σ^{+}_{u}.

The symbol Σ^{+}_{g} refers to one-nucleon states with positive parity ($\frac{1}{2}^{+}$), and Σ^{+}_{u} refers to states with negative parity ($\frac{1}{2}^{-}$).

In this classification we have the following constants of the motion (besides the trivial ones): the nucleon spin s, the total rotational moment $\kappa = \boldsymbol{l} + \mathbf{R}$ (**R** is the vector of the moment of the collective rotation), the projection of the vector of the orbital angular momentum of the nucleon $\boldsymbol{l}$ on the nuclear axis $\Lambda = 0$, and the projection of κ on the nuclear axis $\kappa_{z'} = 0$.

Using the above-mentioned symmetry properties of the wave function, we can in this case easily compute the matrix element (14). We obtain the result

$$\begin{aligned} \frac{\lambda}{mc^2} \left\langle r \frac{\partial U}{\partial r} [(\mathbf{s} \cdot \mathbf{n})(\mathbf{n} \cdot \mathbf{J}) - (\mathbf{s} \cdot \mathbf{J})] \right\rangle &= \left[\frac{1}{4} \left\langle r \frac{\partial U}{\partial r} (n^2_{z'} - n^2_{x'}) \right\rangle \right. \\ &\left. + \langle (\mathbf{s} \cdot \mathbf{J}) \rangle \left\langle r \frac{\partial U}{\partial r} n^2_{x'} \right\rangle - \left\langle (\mathbf{s} \cdot \mathbf{J})\, r \frac{\partial U}{\partial r} \right\rangle \right] \frac{\lambda}{mc^2}. \end{aligned} \tag{19}$$

It follows from the theory of the rotational spectra of nuclei with spin $\frac{1}{2}$, developed by us earlier,[11] that the energies of the rotational levels of these nuclei can be obtained from the Hamiltonian

$$\mathcal{H}_{\text{coll}} = \hbar^2 \varkappa^2 / 2I + \gamma \varkappa \cdot \mathbf{s}, \tag{20}$$

where I is the moment of inertia, β the deformation parameter, and

$$\gamma = \frac{3}{2} \sqrt{\frac{5}{4\pi}} \beta \frac{\hbar^2}{I} \frac{\lambda}{mc^2} \left\langle r \frac{\partial U}{\partial r} (n^2_{z'} - n^2_{x'}) \right\rangle.$$

Finally, we obtain the following expression for the additional magnetic moment of nuclei with spin $\frac{1}{2}$:

$$\begin{aligned} \mu = \frac{e\hbar}{2mc(J+1)} \Big\{ & \frac{\gamma}{(3/2)\sqrt{5/4\pi}\, \beta \hbar^2 / I} [1/4 - 1/3 \langle (\mathbf{s} \cdot \mathbf{J}) \rangle] \\ & - 2/3 \langle (\mathbf{s} \cdot \mathbf{J}) \rangle \left\langle r \frac{\partial U}{\partial r} \right\rangle \frac{\lambda}{mc^2} \Big\}. \end{aligned} \tag{21}$$

Let us apply formula (21) to the calculation of the magnetic moment of Tm^{169} The rotational spectrum corresponds in this case[12] to the Σ^{+}_{u} state, for which $\kappa = 1, 3, 5, \ldots$. For this nucleus $\beta = 0.3$ (reference 13) and $\gamma/(\hbar^2/I) = 0.24$. If we further assume that $\langle r \partial U / \partial r \rangle$ is in this case of the same order of magnitude as for spherical nuclei, and that $g_R = 0.41$, we obtain $\mu_{\text{tot}} = -0.35$ nuclear magnetons for the total magnetic moment of this nucleus. The experimental value for this magnetic moment is $\mu_{exp} = -0.20 \pm 0.05$. The comparison of these two values shows that the magnetic moment of Tm^{169} does not contradict the adopted classification of the ground state of this nucleus.

It is seen from these calculations that the interaction (2) can make a significant contribution to the magnetic moments of spherical and nonspherical nuclei, which must not be neglected.

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