ON A CERTAIN VERSION OF NONLOCAL ELECTROMAGNETIC FIELD THEORY

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A nonlocal theory is considered which corresponds to a modification of the Lienard-Wiechert potential (2). It is shown that if, in this version of the theory, the potential is assumed to satisfy the usual type of equation (3), the consistent relativistic treatment (many-time formalism) leads to self-contradictions already within the framework of the classical theory.

AT various times a number of authors[1-3] investigated one of the versions of nonlocal electromagnetic field theory from different points of view. This version is based on a modification of the form of the Lienard-Wiechert potentials. Instead of the usual way of writing,

$$A_\mu = -e\,(u_\mu(r')\,/\,R_\nu u_\nu(r'))_{R_\nu^2=0} \tag{1}$$

(where u_μ is the four-velocity of the charge, $R_\nu = r_\nu - r'_\nu$ is the difference of the coordinates of the point of observation and the charge), one starts from the following expression:

$$A_\mu = -e\,(u_\mu(r')\,/\,R_\nu u_\nu(r'))_{R_\nu^2+a^2=0}. \tag{2}$$

Here a is a new constant with the dimension of a length. It is further assumed that the potential satisfies the usual equation

$$\Box A_\mu = -\frac{4\pi}{c} j_\mu, \tag{3}$$

which essentially defines the current. In the static case $u_\nu = (p, 0, 0, i)$, and (3) leads to the charge density

$$\rho(\mathbf{r}-\mathbf{r}') = (e/4\pi)\,3a^2/(\mathrm{R}^2+a^2)^{5/2}, \tag{4}$$

i.e., the point charge is spread out. Analogously, the current is spread out in a more complicated way. In this theory the field at the location of the charge is finite, and so is the self energy.

The aim of the present paper is to show that this approach is unsatisfactory, and that self-contradictions arise from the adoption of the usual type of equation for the potential leading to a spreading out of the charge: the classical system of equations for charged particles is internally inconsistent. For the proof we make use of the many-time formulation of the classical electrodynamics of a system of charged particles.[4] We assume that each particle, and the field, has its own time, different from that of the others. The interaction for such a system may be written in the form

$$S = \sum_{n=1}^{N}\left\{\int_0^{t_n^0}[-m_nc^2 + u_\nu(\mathbf{r}_n, t_n) \times \int \rho_n(\mathbf{r}-\mathbf{r}_n)\,A_\nu(\mathbf{r}, t_n)\,d\mathbf{r}]\sqrt{1-\beta_n^2}\,dt_n\right\} + \int_0^{t^0} dt \int\left[-\frac{1}{8\pi}\left(\frac{\partial A_\nu}{\partial x_\mu}\right)^2\right] d^3x. \tag{5}$$

The Hamiltonian of the particles is

$$H_n = \int \rho_n(\mathbf{r}-\mathbf{r}_n)\,\varphi(\mathbf{r}, t_n)\,d\mathbf{r} + c\left[m_n^2c^2 + \left(\mathbf{p}_n - \frac{1}{c}\int\rho_n(\mathbf{r}-\mathbf{r}_n)\,\mathbf{A}(\mathbf{r}, t_n)\,d\mathbf{r}\right)^2\right]^{1/2}, \tag{6}$$

the Hamiltonian of the field is

$$H = \frac{1}{8\pi}\int\left(\sum_{i=1}^{3}\left(\frac{\partial A_\nu}{\partial x_i}\frac{\partial A_\nu}{\partial x_i}\right) + (4\pi)^2c^2\Pi_\nu^2\right)d^3x, \tag{7}$$

where

$$\Pi_\nu = (1/4\pi c^2)\,\partial A_\nu/\partial t.$$

We note that, in the many-time formalism, the electromagnetic field is described by the free Hamiltonian and the corresponding free field equation $\Box A_\mu = 0$. A current at the right hand side of the field equation appears only if the transition to a single time is made.

For the following it is convenient to go over to different canonical field variables. Let

$$A_\nu(\mathbf{r}, t) = V^{-1/2}\sum_k\left[\frac{P_k^\nu}{k}\cos(\mathbf{k}\cdot\mathbf{r}-kct) + cQ_k^\nu\sin(\mathbf{k}\cdot\mathbf{r}-kct)\right],$$

$$4\pi\Pi_\nu(\mathbf{r}, t) = (V^{-1/2}\sum_k\left[\frac{1}{c}P_k^\nu\sin(\mathbf{k}\cdot\mathbf{r}-kct) - Q_k^\nu k\cos(\mathbf{k}\cdot\mathbf{r}-kct)\right]. \tag{8}$$

P_k^ν and Q_k^ν are the new canonical variables:

$$[P_k^\nu, Q_{k'}^\mu] = 4\pi\delta_{\mu\nu}\delta_{kk'}, \ [P_k^\nu, P_{k'}^\mu] = 0, \ [Q_k^\nu, Q_{k'}^\mu] = 0. \quad (9)$$

Expressions (8) explicitly give the time dependence of P_k^ν and Q_k^ν.

The field Hamiltonian in the new canonical variables has the form

$$H = \frac{1}{8\pi}\sum_k [(P_k^\nu)^2 + c^2k^2(Q_k^\nu)^2]. \quad (10)$$

The terms in the Hamiltonian for the particles which depend on the field are conveniently rewritten in the following way:

$$\int \rho_n(\mathbf{r}-\mathbf{r}_n) A_\nu(\mathbf{r}, t_n)\, d\mathbf{r}$$

$$= \int \rho_n(\mathbf{r}-\mathbf{r}_n)\Big[V^{-1/2}\sum_k\Big(\frac{P_k^\nu}{k}\cos(\mathbf{k}\cdot\mathbf{r} - kct_n) + cQ_k^\nu\sin(\mathbf{k}\cdot\mathbf{r} - kct_n)\Big)\Big]d\mathbf{r}$$

$$= \int \rho_n(\mathbf{R})\Big[V^{-1/2}\sum_k\Big(\frac{P_k^\nu}{k}\cos(\mathbf{k}\cdot\mathbf{R} + \mathbf{k}\cdot\mathbf{r}_n - kct_n) + cQ_k^\nu\sin(\mathbf{k}\cdot\mathbf{R} + \mathbf{k}\cdot\mathbf{r}_n - kct_n)\Big)\Big]d\mathbf{R}.$$

Since $\rho_n(\mathbf{R})$ depends only on R^2, this expression becomes

$$V^{-1/2}\sum_k\Big[\Big(\int\rho_n(R)\cos\mathbf{k}\cdot\mathbf{R}\,d\mathbf{R}\Big)\Big(\frac{P_k^\nu}{k}\cos(\mathbf{k}\cdot\mathbf{r}_n - kct_n) + cQ_k^\nu\sin(\mathbf{k}\cdot\mathbf{r}_n - kct_n)\Big)\Big]$$

$$= V^{-1/2} e_n \sum_k f(k) \times\Big(\frac{P_k^\nu}{k}\cos(\mathbf{k}\cdot\mathbf{r}_n - kct_n) + cQ_k^\nu\sin(\mathbf{k}\cdot\mathbf{r}_n - kct_n)\Big), \quad (11)$$

where

$$f(k) = \frac{1}{e_n}\int\rho_n(R)\cos\mathbf{k}\cdot\mathbf{R}\,d\mathbf{R}.$$

The Hamilton-Jacobi equations for our system of charged particles are written in the following form:

$$\partial S(t_1\ldots t_N)/\partial t_n = -H_n(\mathbf{r}_n, t_n, \partial S/\partial\mathbf{r}_n, Q_k^\nu, \partial S/\partial Q_k^\nu),$$

$$(n = 1\ldots N). \quad (13)$$

The condition of consistency for such a system of equations is (see, e.g., reference 5)

$$(\partial^2/\partial t_{n'}\partial t_n - \partial^2/\partial t_n\partial t_{n'})S(t_1\ldots t_N) = 0. \quad (14)$$

According to (13) this leads to

$$\partial H_n\Big(\mathbf{r}_n, t_n, \frac{\partial S}{\partial \mathbf{r}_n}, Q_k^\nu, \frac{\partial S}{\partial Q_k^\nu}\Big)\Big/\partial t_{n'} - \partial H_{n'}\Big(\mathbf{r}_{n'}, t_{n'}, \frac{\partial S}{\partial\mathbf{r}_{n'}}, Q_k^\mu, \frac{\partial S}{\partial Q_k^\mu}\Big)\Big/\partial t_n = 0$$

or

$$\frac{\partial H_n}{\partial(\partial S/\partial\mathbf{r}_n)}\frac{\partial^2 S}{\partial t_{n'}\partial\mathbf{r}_n} + \frac{\partial H_n}{\partial(\partial S/\partial Q_k^\nu)}\frac{\partial^2 S}{\partial t_{n'}\partial Q_k^\nu} - \frac{\partial H_{n'}}{\partial(\partial S/\partial\mathbf{r}_{n'})}\frac{\partial^2 S}{\partial t_n\partial\mathbf{r}_{n'}} - \frac{\partial H_{n'}}{\partial(\partial S/\partial Q_k^\mu)}\frac{\partial^2 S}{\partial t_n\partial Q_k^\mu} = 0.$$

We note that $\partial S/\partial\mathbf{r}_n$ and $\partial S/\partial Q_k^\nu$ correspond to the canonical momenta of the particles and the field and that the order of differentiation with respect to the time and to the coordinates (of the particles and the field) can be interchanged if the condition of consistency is satisfied. The last expression can then be rewritten in form

$$\frac{\partial H_n}{\partial\mathbf{p}_n}\frac{\partial H_{n'}}{\partial\mathbf{r}_n} + \frac{\partial H_n}{\partial P_k^\nu}\frac{\partial H_{n'}}{\partial Q_k^\nu} - \frac{\partial H_{n'}}{\partial\mathbf{p}_{n'}}\frac{\partial H_n}{\partial\mathbf{r}_{n'}} - \frac{\partial H_{n'}}{\partial P_k^\mu}\frac{\partial H_n}{\partial Q_k^\mu} = 0.$$

Since always $\partial H_n/\partial\mathbf{p}_{n'} = \partial H_n/\partial\mathbf{r}_{n'} = 0$ $(n \neq n')$, we finally get instead of (14)

$$[H_n, H_{n'}] = 0. \quad (15)$$

Here $[H_n, H_{n'}]$ is the classical Poisson bracket. It is clear from the foregoing that in our case the Poisson bracket is calculated from the canonical field variables only.

From (6) and (11) we have

$$[H_n, H_{n'}] = \frac{e_n e_{n'} c}{V}\sum_k\Big\{\frac{f^2(k)}{k}[\cos(\mathbf{k}\cdot\mathbf{r}_n - kct_n) \times\sin(\mathbf{k}\cdot\mathbf{r}_{n'} - kct_{n'}) - \sin(\mathbf{k}\cdot\mathbf{r}_n - kct_n)\cos(\mathbf{k}\cdot\mathbf{r}_{n'} - kct_{n'})]$$

$$\times\Big(\frac{\delta_{\nu 4}}{i} - \sum_{\alpha=1}^{3}\beta_\alpha^n\delta_{\nu\alpha}\Big)\Big(\frac{\delta_{\nu 4}}{i} - \sum_{\gamma=1}^{3}\beta_\gamma^{n'}\delta_{\nu\gamma}\Big)\Big\} = \frac{e_n e_{n'} c}{V}(1 - \beta_n\beta_{n'}) \sum_k\Big\{\frac{f^2(k)}{k}\sin[\mathbf{k}\cdot(\mathbf{r}_n - \mathbf{r}_{n'}) - kc(t_n - t_{n'})]\Big\}$$

$$= \frac{1}{(2\pi)^3}e_n e_{n'} c(1 - \beta_n\beta_{n'}) \times\int\frac{f^2(k)}{k}\sin[\mathbf{k}\cdot(\mathbf{r}_n - \mathbf{r}_{n'}) - kc(t_n - t_{n'})]\,d^3k.$$

Finally,

$$[H_n, H_{n'}] = \frac{e_n e_{n'} c}{(2\pi)^3}(1 - \beta_n\beta_{n'}) \times\int\frac{f^2(k)}{k}\sin[\mathbf{k}\cdot(\mathbf{r}_n - \mathbf{r}_{n'}) - kc(t_n - t_{n'})]d^3k. \quad (16)$$

In the case of a point charge, $f(k) = 1$, and the expression

$$-(2\pi)^{-3}\int\sin[\mathbf{k}\cdot(\mathbf{r}_n - \mathbf{r}_{n'}) - kc(t_n - t_{n'})]\frac{d^3k}{kc}$$

$$= (2\pi)^{-3}\int\cos\mathbf{k}\cdot(\mathbf{r}_n - \mathbf{r}_{n'})\sin kc(t_n - t_{n'})\frac{d^3k}{kc} = D(x_n - x_{n'})$$

agrees with the well-known commutator function. For point particles the conditions of consistency are therefore satisfied everywhere outside the light cone and hence, what is especially important, in the space-like region.

In our case $f(k)$ is equal to unity, and

$$f(k) = \frac{3a^2}{4\pi}\int \frac{\cos \mathbf{k}\cdot\mathbf{R}\, d\mathbf{R}}{(R^2+a^2)^{5/2}} = 3a^2\int_0^\infty \frac{\sin kR}{k(R^2+a^2)^{5/2}} R dR$$
$$= a^2 \int_0^\infty \frac{\cos kR}{(R^2+a^2)^{3/2}} dR = kaK_1(ka). \qquad (17)$$

Here K_1 is the Hankel function with imaginary argument. Since $f(k)$ depends only on the absolute value of k, the integral expression in (16) becomes

$$-\frac{4\pi}{|\mathbf{r}_n - \mathbf{r}_{n'}|}\int_0^\infty f^2(k) \sin k |\mathbf{r}_n - \mathbf{r}_{n'}| \sin kc(t_n - t_{n'})\, dk$$
$$= \frac{2\pi a^2}{|\mathbf{r}_n - \mathbf{r}_{n'}|}\int_0^\infty k^2 K_1^2(ka) \{\cos [k(|\mathbf{r}_n - \mathbf{r}_{n'}| + c(t_n - t_{n'}))]$$
$$- \cos [k(|\mathbf{r}_n - \mathbf{r}_{n'}| - c(t_n - t_{n'}))]\}\, dk$$

and therefore

$$[H_n, H_{n'}] = \frac{e_n e_{n'} c}{4\pi^2 |\mathbf{r}_n - \mathbf{r}_{n'}|}(1 - \beta_n\beta_{n'})\, a^2$$
$$\times \int_0^\infty k^2 K_1^2(ka) \{\cos [k(|\mathbf{r}_n - \mathbf{r}_{n'}|$$
$$+ c(t_n - t_{n'}))] - \cos [k(|\mathbf{r}_n - \mathbf{r}_{n'}| - c(t_n - t_{n'}))]\}\, dk. \qquad (18)$$

The integration cannot be performed analytically. The integral can, however, be evaluated approximately near the light cone. For this purpose we make use of the integral representation

$$kK_1(ka) = a\int_0^\infty \frac{\cos kR}{(R^2+a^2)^{3/2}}\, dR$$

and change the order of integration in the integral (18). Using the notation

$$|\mathbf{r}_n - \mathbf{r}_{n'}| + c(t_n - t_{n'}) = \Delta_1, \quad |\mathbf{r}_n - \mathbf{r}_{n'}| - c(t_n - t_{n'}) = \Delta_2;$$

we have

$$\int_0^\infty k^2 K_1^2(ka) \Big[\cos k\Delta_1 - \cos k\Delta_2\Big]\, dk$$
$$= \frac{a^2\pi}{4}\int_0^\infty \frac{dR}{(R^2+a^2)^{3/2}} \{[(\Delta_1 + R)^2 + a^2]^{-3/2}$$
$$+ [(\Delta_1 - R)^2 + a^2]^{-3/2} - [(\Delta_2 + R)^2 + a^2]^{-3/2}$$
$$- [(\Delta_2 - R)^2 + a^2]^{-3/2}\}. \qquad (19)$$

We note that Δ_1/a may be quite large for $\Delta_2/a \ll 1$ and vice versa, for $|\mathbf{r}_n - \mathbf{r}_{n'}|$ and $c(t_n - t_{n'})$ may be taken as large as one pleases. Let us assume that $\Delta_2/a \ll 1$, and let us neglect the terms with Δ_1 in view of what has just been said. Making a series expansion, we then obtain from (19)

$$-(3\pi^2/32a^3)(1 - 15\Delta_2^2/32a^2).$$

With $\Delta_2/a \ll 1$, $\Delta_1/a \gg 1$ we then have

$$[H_n, H_{n'}] \approx -\frac{3e_n e_{n'} c}{128a|\mathbf{r}_n - \mathbf{r}_{n'}|}(1 - \beta_n\beta_{n'})\Big(1 - \frac{15\Delta_2^2}{32a^2}\Big)$$
$$= -0{,}023\frac{e_n e_{n'} c}{a|\mathbf{r}_n - \mathbf{r}_{n'}|}(1 - \beta_n\beta_{n'})\Big(1 - \frac{15\Delta_2^2}{32a^2}\Big),$$

i.e., this expression is different from zero on the cone and near the cone in the space-like region.

An analogous calculation can be carried out for the case $\Delta_1/a \ll 1$, $\Delta_2/a \gg 1$. Furthermore, we calculated the integral numerically for the value $\Delta_2/a = 10$, i.e., in a region which is comparatively far from the cone (Δ_1/a was assumed to be sufficiently large, so that the terms with Δ_1 could be omitted). The result is

$$[H_n, H_{n'}] \approx -0.0006\frac{e_n e_{n'} c}{a|\mathbf{r}_n - \mathbf{r}_{n'}|}(1 - \beta_n\beta_{n'}).$$

Hence $[H_n, H_{n'}]$ is different from zero on the light cone and everywhere in the space-like region, and, although it decreases fast with the distance from the cone, it nowhere reduces to zero, except at infinity ($\Delta_1/a = \infty$, $\Delta_2/a = \infty$).*

In other words, the conditions of consistency of the system (13) are not fulfilled in this version of the theory.

In the process of quantization the classical Poisson bracket goes over into the commutator of the corresponding quantities:

$$[H_n, H_{n'}]_{\rm cl} \to \frac{i}{h}[\hat{H}_n, \hat{H}_{n'}]_{\rm qu} = \frac{i}{h}[\hat{H}_n\hat{H}_{n'} - \hat{H}_{n'}\hat{H}_n].$$

In the case of point particles this leads to (with $\beta_n \to \alpha_n$, $\beta_{n'} \to \alpha_{n'}$, where α_n, $\alpha_{n'}$ are the Dirac matrices)

$$[\hat{H}_n, \hat{H}_{n'}]_{\rm cl} = -ihc^2 e_n e_{n'} (\boldsymbol{\alpha}_n \alpha_{n'} - 1)\, D(x_n - x_{n'}).$$

the well-known expression for the Bloch condition.[6] The failure of $[H_n, H_{n'}]_{\rm cl}$ to reduce to zero in the space-like region in this nonlocal version of the theory must lead to similar difficulties in the quantum region.

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154

*This may also be seen qualitatively from expression (18), for as Δ_1/a and Δ_2/a increase, the cosine begins to oscillate rapidly, thus making the value of the integral smaller.