BEHAVIOR OF A CONDUCTING GASEOUS SPHERE IN A QUASI-STATIONARY ELECTROMAGNETIC FIELD

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The stability of a homogeneous plasma sphere of infinite conductivity in an external quasi-stationary electromagnetic field is investigated by perturbation-theory methods.

In recent years a number of papers have appeared which consider the equilibrium states of an isolated plasma in an external electromagnetic field (cf. references 1 — 3). Equilibrium between the field and the bound plasma configuration obtains by virtue of equilibration of the electrodynamic and hydrostatic forces. The behavior of an equilibrium system of this type, in particular as it pertains to problems of stability, are of great interest.[4] The stability of a plasma is also of great interest in connection with methods of radiation acceleration of charged-particle bunches.[5]

In the present work perturbation theory is used to investigate the stability of a conducting sphere comprising a completely ionized gas which is located in an external quasi-stationary field.

It is assumed that the electrical conductivity of the plasma is infinite. The plasma sphere itself is considered a uniform adiabatic system which obeys the equation of state of an ideal gas and is characterized by one external parameter — the radiation pressure at the surface (gravitational forces are neglected).

PLASMA SPHERE IN A UNIFORM FIELD

For simplicity we first consider a gaseous sphere of infinite conductivity located in a quasi-stationary spatially uniform electromagnetic field; the field components are given functions of time:

$$\mathbf{E} = \{E_x, E_y, E_z\} = E_0\{\exp(i\Omega_{1e}t), \exp(i\Omega_{2e}t), \exp(i\Omega_{3e}t)\},$$
$$\mathbf{H} = \{H_x, H_y, H_z\}$$
$$= H_0\{\exp(i\Omega_{1m}t), \exp(i\Omega_{2m}t), \exp(i\Omega_{3m}t)\}, \quad (1)$$

where all frequencies Ω differ from each other. The effective amplitude of this alternating (rotating) field is independent of direction; thus the mean pressure is uniform everywhere over the surface of the sphere and equilibrium obtains for the spherical shape. Obviously the field in (1) can only be an approximation to any actual electromagnetic fields of this form and is actually a superposition of standing or traveling waves which are polarized in various directions. Actually, if the dimensions of the bunch are small compared with the wavelength (quasi-stationary case) the inhomogeneity and wave properties of the field can be neglected in considering all problems except those which relate to the behavior of the bunch as a whole (these will be considered separately).

The investigation of the stability of a plasma sphere in the field given by (1) is carried out by means of an energy approach. Since the quasi-stationary conditions are satisfied we can determine the electromagnetic energy of a bunch in the external field (1) starting from well-known formulas of electrostatics magnetostatics.* An ideally conducting plasma bunch which cannot be penetrated by an alternating electromagnetic field ($\mathbf{E}^{\text{int}} = 0$, $\mathbf{B}^{\text{int}} = \mu$, $\mathbf{H}^{\text{int}} = 0$) may be considered phenomenologically as a body with infinite dielectric susceptibility $\epsilon \to \infty$ and zero magnetic permeability $\mu = 0$. Then the potential energy in the external field at any instant of time is given by

$$U(t) = \frac{1}{8\pi}\int(\mathbf{H}^{\text{int}}\cdot\mathbf{H} - \mathbf{D}^{\text{int}}\cdot\mathbf{E})\,dv. \quad (2)$$

In Eq. (2) the integration is carried out over the volume of the bunch while the magnetic field $\mathbf{H}^{\text{int}} = -\operatorname{grad}\psi_m^{\text{int}}$ and the electric induction $\mathbf{D}^{\text{int}} = -\operatorname{grad}\psi_e^{\text{int}}$ inside the bunch are found from the solutions of the Laplace equation

$$\Delta\psi_m = 0, \quad \Delta\psi_e = 0 \quad (3)$$

for potentials which satisfy the boundary conditions

$$\psi_m^{\text{int}} = \psi_m^{\text{ext}}, \quad \psi_e^{\text{ext}} = 0,$$
$$\partial\psi_m^{\text{ext}}/\partial n = 0, \quad \partial\psi_e^{\text{int}}/\partial n = \partial\psi_e^{\text{ext}}/\partial n \quad (4)$$

at the surface of the bunch. The basic problem

*Our attention was directed to this fact by M. L. Levin.

now becomes the determination of the mean potential energy of the bunch in a given external field as a function of the variables which characterize arbitrarily small deformation of the sphere. Let the surface be characterized by a function $R(\vartheta, \varphi)$ which characterizes the distance from the center of the sphere to a point on the surface given by the angles ϑ and φ in a spherical system of coordinates; the origin is taken at the center of the sphere. $R(\vartheta, \varphi)$ can be expanded in spherical functions:

$$R(\vartheta, \varphi) = R_0\Big[1 + \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \alpha_{lm} Y_l^m(\vartheta, \varphi)\Big].$$

The expansion coefficients α_{lm} are generalized coordinates; at any instant of time these determine uniquely a definite arbitrary configuration of the surface of the bunch. In analyzing the motion in the neighborhood of the equilibrium configuration we limit ourselves to weak perturbations of the sphere, in which case $\alpha_{lm} \ll 1$.

In accordance with perturbation theory, the solutions of Eq. (3) which satisfy (4) at the boundary of the bunch are sought as a series in increasing powers of the small deformation parameters α_{lm}. We limit ourselves to second-order perturbations and neglect intermediate contributions, giving only the final result. The time average of the electromagnetic potential energy of an infinitely conducting plasma bunch in a quasi-stationary external field (1) is given by the expression

$$\bar{U} = \frac{9H_0^2}{32\pi} V_0 \Big\{\Big(1 - 2\frac{E_0^2}{H_0^2}\Big)\frac{V}{V_0} + \sum_{l=2}^{\infty}\sum_{m=-l}^{l} \frac{3}{4\pi} N_{lm} \frac{(2l-1)(l-1)}{2(2l+1)} \times \Big[1 - \frac{4}{3}\frac{(l+1)l}{l-1}\frac{E_0^2}{H_0^2}\Big]\alpha_{lm}^2\Big\}, \tag{5}$$

where $V_0 = 4\pi R_0^3/3$ is the equilibrium volume of the bunch, N_{lm} is the index of the spherical function $Y_l^m(\vartheta, \varphi)$. The summation in (5) starts with $l = 2$. When $l = 0$ only the volume of the sphere is changed and the deformation corresponding to $l = 1$ is associated with the displacement of the sphere as a whole, which makes no contribution to the energy in the case of a uniform field.

Writing the deformation parameters α_{lm} in the form of functionals

$$\alpha_{lm} = N_{lm}^{-1}\int_0^{2\pi}\int_0^{\pi} (R/R_0)\, Y_l^m \sin\vartheta\, d\vartheta\, d\varphi \approx N_{lm}^{-1}R_0^{-3}\int_S R Y_l^m d\sigma,$$

where the integration is carried out over the surface of the bunch, we find the mean pressure as a variational derivative of the potential energy:

$$P(\vartheta, \varphi) = \frac{\delta\bar{U}}{\delta R} = \frac{9H_0^2}{32\pi}\Big\{\Big(1 - 2\frac{E_0^2}{H_0^2}\Big) + \sum_{l=2}^{\infty}\sum_{m=-l}^{l} \frac{(2l-1)(l-1)}{2l+1} \times \Big[1 - \frac{4}{3}\frac{(l+1)l}{l-1}\frac{E_0^2}{H_0^2}\Big]\alpha_{lm}Y_l^m(\vartheta, \varphi)\Big\}. \tag{6}$$

The term in Eq. (6) which is independent of deformation is the constant electromagnetic pressure

$$P_0 = \frac{9H_0^2}{32\pi}\Big(1 - 2\frac{E_0^2}{H_0^2}\Big),$$

which is directed along the normal to the surface (inward or outward). Adding to Eq. (5) the expression for potential energy corresponding to the work performed by the gas in the adiabatic process we obtain the total potential energy of a uniform plasma bunch near equilibrium:

$$\bar{U} + W_g = \frac{9H_0^2}{32\pi} V_0 \Big\{\Big(1 - 2\frac{E_0^2}{H_0^2}\Big)\Big[1 + \frac{1}{\gamma - 1} + \frac{\gamma}{2}\Big(\frac{V - V_0}{V_0}\Big)^2\Big] + \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\frac{3}{4\pi}N_{lm}\frac{(2l-1)(l-1)}{2(2l+1)}\Big[1 - \frac{4}{3}\frac{(l+1)l}{l-1}\frac{E_0^2}{H_0^2}\Big]\alpha_{lm}^2\Big\}, \tag{7}$$

where $\gamma = c_p/c_v$ is the ratio of the specific heats. An analysis of the last expression yields certain conclusions regarding the behavior of a conducting gaseous sphere in electromagnetic fields such as those described by Eq. (1).

We first consider the effect of a uniform quasi-stationary magnetic field (a field of this type was used in the work reported by Knox[3]). In Eq. (7) we set $E_0 = 0$, thereby obtaining a situation of minimum potential energy for $V = V_0$; then all the $\alpha_{lm} = 0$. Whence it follows that a spherical bunch of radius R_0 is stable against an arbitrary small deformation.

In the case in which only an electric field operates the surface of the sphere is subject to forces of negative electric pressure, $P_0 < 0$; thus a bunch in a void cannot be in equilibrium.

In the general case of superposition of electric and magnetic fields a stable volume for a bunch in a void is possible only if the fixed radiation pressure is positive, $P_0 > 0$, i.e., the following relation must be satisfied: $H_0^2 > 2E_0^2$. The nature of the stability with respect to various deformations is determined by the sign of the quantity

$$\eta_l = 1 - \frac{4}{3}\frac{(l+1)l}{(l-1)}\frac{E_0^2}{H_0^2},$$

which depends on the ratio of the electric and magnetic field amplitudes and on the deformation index l. A bunch is stable against elementary deformation characterized by the indices l and m if $\eta > 0$ and is unstable if $\eta < 0$. In particular, in order for a spherical bunch to be stable against a simple ellipsoidal deformation[6] ($l = 2$) the inequality $H_0^2 > 8E_0^2$ must be satisfied (the same applies for $l = 3$). As the value of l increases, denoting more complicated deformations, the relation between the electric and magnetic fields becomes more stringent; when $l \gg 1$ this relation becomes $H_0^2 > 4E_0^2 l/3$. Whence it follows that in the presence of an electric field an ideally conducting bunch can be stable only against deformations which are characterized by a finite number of first surface harmonics satisfying the relation

$$\eta > 0. \tag{8}$$

The nature of the stability criterion (8) is intimately related to the basic assumptions made at the beginning of this paper concerning the ideal electrical conductivity of the plasma. As applied to a real bunch this supposition is valid as long as the wavelength $\lambda = 2\pi R_0/l$ of the corresponding surface harmonics is much greater than the skin depth d.

STABILITY IN A QUASI-UNIFORM FIELD

Above we have investigated the stability of a plasma bunch in an idealized spatially uniform field (1). In treating actual cases we must take account of the small inhomogeneity in the external field.

We consider a conducting plasma sphere in an external quasi-uniform field which may conveniently be written as follows:

(1) Quasi-uniform electric field

$$\mathbf{E}_e(\mathbf{r}) = \mathbf{E}_0 + \mathbf{E}_{1e}(\mathbf{r}), \quad \mathbf{H}_e(\mathbf{r}) = \mathbf{H}_{1e}(\mathbf{r}); \tag{9}$$

(2) Quasi-uniform magnetic field

$$\mathbf{H}_m(\mathbf{r}) = \mathbf{H}_0 + \mathbf{H}_{1m}(\mathbf{r}), \quad \mathbf{E}_m(\mathbf{r}) = \mathbf{E}_{1m}(\mathbf{r}), \tag{10}$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are given by (1) as before and the small fields denoted by the subscript "1" are not considered in detail but merely characterize the small inhomogeneity of the applied field. We now use Eqs. (9) and (10) in place of (1) in the original expression for the potential energy. Taking account of the small variation of the external field over the bunch, we obtain an approximate expression for the mean potential energy in a quasi-uniform field:

$$\overline{U}(\mathbf{r}, V, \alpha) = \overline{U}(\mathbf{r}) + \overline{U}(V, \alpha),$$

where $\overline{U}(V, \alpha)$ corresponds to the formula derived earlier (7), while

$$\overline{U}(\mathbf{r}) = \frac{3}{16\pi} V_0 [2(\mathbf{H}_0 \cdot \mathbf{H}_{1m} - 2\mathbf{E}_0 \cdot \mathbf{E}_{1e}) + (H_{1e}^2 - 2E_{1m}^2) + (H_{1m}^2 - 2E_{1e}^2)]_{\mathrm{av}}$$

depends on the position of the bunch as a whole in the quasi-uniform field ($\mathbf{r}$ is the relative coordinate of the center of the bunch). The forces which act on a spherical bunch, at the center of which the amplitudes of the external fields reach their maximum values E_0 and H_0, vanish when integrated over the volume; thus, the bunch as a whole is in a state of quilibrium in the inhomogeneous field. The nature of the stability is determined by the sign of $\overline{U}(\mathbf{r})$ in the neighborhood of equilibrium.

We may illustrate the application of this analysis by a simple example, using the superposition of six standing plane waves with different wave numbers $k = \Omega/c$ in an appropriate configuration. It is not difficult to show that in this quasi-uniform wave field

$$\overline{U}(\mathbf{r}) = \frac{9H_0^2}{32\pi} V_0 \left[\left(\frac{E_0^2}{H_0^2} k_{2e}^2 - k_{2m}^2 \right) x^2 + \left(\frac{E_0^2}{H_0^2} k_{3e}^2 - k_{3m}^2 \right) y^2 + \left(\frac{E_0^2}{H_0^2} k_{1e}^2 - k_{1m}^2 \right) z^2 \right],$$

where x, y, z denotes the departure of the center of the sphere from the location of the common antinode of the standing waves. From this follows the stability criterion:

$$H_0^2/E_0^2 < (k_{je}/k_{jm})^2, \quad j = 1, 2, 3. \tag{11}$$

Comparing Eq. (11) with the criterion for internal stability of a highly conducting bunch ($d \ll R_0$) with respect to a change of volume and shape:

$$H_0^2/E_0^2 > 4(l+1)l/3(l-1), \quad 2 \leqslant l < \pi R_0/2d,$$

we see that these inequalities are incompatible with respect to field amplitudes. However, since the first relation involves amplitudes which are related to the wave properties of the fields while the second involves only quasi-static properties, over a wide region of wave numbers for which

$$k_{je}^2/k_{jm}^2 > 4(l+1)l/3(l-1),$$

both inequalities can be satisfied simultaneously and a spherical bunch is characterized by stability with respect to all simple types of small perturbations of volume, shape, and position in an external field.

Similar results can be obtained in fields which are more complicated than plane-wave fields; for example we may consider fields which are formed by an appropriate configuration of axially symmetric electric and magnetic waves.

In conclusion we may point out that in a general consideration of bunches of charged particles one must inevitably encounter difficulties which stem from the fact that the system has a limited number of degrees of freedom. The rather crude phenomenological model used in the present paper does, however, indicate the basic features of the behavior of plasma in quasi-stationary fields.

Inasmuch as the purpose of the present note was to investigate the stability of a highly conducting gaseous sphere in an external field we have limited ourselves to small deformation of the surface and have not considered transient effects.

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[1] M. Krushkal and M. Schwarzschild, Proc. Roy. Soc. (London), **A223**, 348 (1954); V. D. Shafranov, Атомная энергия (Atomic Energy) **5**, 38 (1956).

[2] V. D. Shafranov, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 710 (1957), Soviet Phys. JETP **6**, 545 (1958).

[3] F. B. Knox, Australian J. Phys. **10**, 221, 565 (1957).

[4] V. I. Veksler, L. M. Kovrizhnykh, M. S. Rabinovich and V. V. Yankov, Possibility of Using Electromagnetic Waves for Stabilization of Plasma Bunches, Report of the Inst. Phys. Acad. Sci. August, 1956.

[5] V. I. Veksler, Атомная энергия (Atomic Energy) **2**, 427 (1957).

[6] V. V. Yankov, The Stability of a Quasi-Ellipsoidal Plasma Bunch in an Alternating Electromagnetic Field, Report of the Inst. Phys. Acad. Sci., 1957.

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