CYCLIC MOTION OF CHARGED PARTICLES IN AN ELECTRIC FIELD

A. A. KOLOMENSKII and FONG SOY-CEN

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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Expressions are obtained for the cyclic motion of charged particles in an electric guide field. This field can be produced by a system of appropriate lenses (weak-focusing or strong-focusing). Resonance acceleration is considered; in particular, we consider phase stability (phase focusing). This is analogous to the usual phase focusing in magnetic fields. An investigation is made of the effect of electromagnetic radiation (including quantum fluctuations) on the motion of electrons in the electric field. The case in which part of the particle trajectory is in an electric field and part in a magnetic field is also treated.

THE cyclic motion of charged particles in various magnetic-field configurations has been considered at great length in the literature. This work pertains chiefly to accelerators, microwave generators, isotope separation, and so on. However, up to now no investigations have been made of the cyclic motion of charged particles in electric fields, although this motion is characterized by a number of interesting features. We may note that various systems of electric lenses can be used and, indeed, have been used (cf. reference 1) in analog models of large magnetic accelerators.* Problems associated with the radiation of relativistic electrons moving in magnetic fields have been considered in a number of papers. From the theoretical point of view it is of interest to compare these results with those obtained when radiation effects are taken into account in analyzing the motion of electrons in an electric guide field. It is also of interest to extend the analysis of phase focusing to the case of cyclic motion in electric fields; phase focusing has, so far, been considered only in connection with the motion of particles in magnetic fields.

1. MOTION OF PARTICLES IN AN AXIALLY SYMMETRIC ELECTRIC FIELD, NEGLECTING RADIATION EFFECTS

We consider a circle of radius r_s in an axially symmetric electric field characterized by the plane of symmetry, $z = 0$. In the first approximation the components of this field, which satisfy the conditions for vanishing divergence, can be written in the form

$$\mathscr{E}_r(r) \approx \mathscr{E}_0(r_s)(1 - n_e x/r_s),\qquad \mathscr{E}_z(z) \approx -\mathscr{E}_0(r_s)(1 - n_e) z/r_s,\quad x = r - r_s. \tag{1.1}$$

The parameter $n_e = -(r_s/\mathscr{E}_0)\,\partial\mathscr{E}/\partial r$, which is introduced in analogy with the customary magnetic field index, n, can be called the electric field index. In a cyclindrical coordinate system, the equations of motion of a charged particle in a field described by (1.1) and an auxiliary accelerating field $\mathscr{E}_0$, are of the form

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\theta}^2 - e\mathscr{E}_0\left(1 - n_e\frac{x}{r_s}\right), \tag{1.2}$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = eV_\theta, \tag{1.3}$$

$$\frac{d}{dt}(m\dot{z}) = e\mathscr{E}_0(1 - n_e)\frac{z}{r_s}. \tag{1.4}$$

For resonance acceleration

$$V_\theta \approx \frac{V_0}{2\pi}\cos\left[\int_0^t \omega_0 dt - q\theta\right], \tag{1.5}$$

where $q = \omega_0/\omega$ is the frequency multiplication factor and ω_0 and V_0 are the frequency and amplitude of the voltage. In an induction accelerator

$$V_0 \approx (1/2\pi c)\,\partial\Phi/\partial t, \tag{1.6}$$

where Φ is the magnetic flux through the orbit.

The following relations hold in the equilibrium orbit

$$\omega = \frac{e\mathscr{E}_0 c}{E_0}\left[\sqrt{\frac{1}{4} + \left(\frac{E_0}{e\mathscr{E}_0 r_s}\right)^2} - \frac{1}{2}\right]^{1/2},\quad \frac{\omega}{\omega} = \frac{1-\beta_s^2}{\beta_s^2}\frac{\dot{E}_s}{E_s}, \tag{1.7}$$

where E is the total energy of the particle and

*The practical application of electric guide fields in accelerators is hindered by the fact that the force exerted on a particle by an electric field (expressed in v/cm) is 300 β times smaller than the force exerted by a magnetic field (measured in oersteds). Yet electric fields are used widely in the auxiliary ion and electron optical systems for injection and extraction of particles in large accelerators. Electric fields can be in electron models of proton accelerators.

the subscript s denotes equilibrium quantities.

The motion is analyzed in the manner usually employed in an axially symmetric magnetic accelerator. In particular, it is possible to introduce the notion of betatron (fast) oscillations (radial and vertical) and synchrotron (slow) oscillations. There is, however, an important qualitative difference between the two cases; this distinction can be understood if one assumes that both guide fields remain constant in time. In the magnetic case the force that acts on the particle is perpendicular to the velocity and the particle energy is not changed. In the case of motion in an electric field the force which acts on a particle which executes oscillations is not always perpendicular to the direction of motion. Hence, although the time average of the particle energy remains constant, the energy itself varies with time. We can expand all quantities about the equilibrium orbit. Considering first-order terms [we first consider the case of resonance acceleration, Eq. (1.5)] and introducing the relations

$$\frac{\Delta E}{E_s} = \frac{\beta_s^2}{1-\beta_s^2}\left[\frac{x}{r_s} - \frac{\dot{\varphi}}{q\omega}\right], \qquad (1.8)$$

(where φ is the particle phase with respect to the accelerating field), we have from Eqs. (1.2) to (1.4), in the linear approximation,

$$\ddot{x} + \frac{\dot{E}_s}{E_s}\dot{x} + \omega^2(1-n_e)x = \frac{\Delta E}{E_s}\frac{c^2}{r_s}(2-\beta_s^2), \qquad (1.9)$$

$$\left(\frac{\Delta \dot{E}}{E_s}\right) = \left(\frac{\dot{\omega}}{\omega} - \frac{\dot{E}_s}{E_s}\right)\frac{\Delta E}{E_s} - \beta_s^2\left(\frac{\dot{E}_s}{E_s} + \frac{\dot{\omega}}{\omega}\right)\frac{x}{r_s} - \beta_s^2\frac{\dot{x}}{r_s} - \frac{eV_0\omega}{2\pi E_s}\sin\varphi_s\Delta\varphi, \qquad (1.10)$$

$$\ddot{z} + \frac{\dot{E}_s}{E_s}\dot{z} + \omega^2(n_e-1)z = 0. \qquad (1.11)$$

The change of energy in the electric guide field is given by the term $-\beta_s^2\dot{x}/r_s$ in Eq. (1.10). The radial betatron and synchrotron oscillations are given by the equations

$$\ddot{x} + \frac{\dot{E}_s}{E_s}\dot{x} + \omega^2(3-n_e-\beta_s^2)x = 0, \qquad (1.12)$$

$$\frac{d}{dt}\left(\frac{E_s}{q\omega^2 K_1}\dot{\varphi}\right) - \frac{eV_0}{2\pi}\cos\varphi = -\frac{eV_0}{2\pi}\cos\varphi_s, \qquad (1.13)$$

where

$$eV_0\cos\varphi_s = \frac{2\pi r_s\mathcal{E}_0}{\omega(1+E_0^2/E_s^2)}, \quad K_1 = \frac{1+n_e(1-\beta_s^2)}{(3-n_e-\beta_s^2)\beta_s^2}. \qquad (1.14)$$

The betatron oscillations in the vertical and radial directions are characterized by the following frequencies respectively:

$$\omega_z = \omega\sqrt{n_e-1}, \quad \omega_r = \omega\sqrt{3-n_e-\beta_s^2}. \qquad (1.15)$$

For stability the following condition must be satisfied:

$$3-\beta_s^2 > n_e > 1. \qquad (1.16)$$

The width of the n_e stability region depends on the particle velocity: when $\beta = 0$ this region is a maximum, when $\beta = 1$ it is a minimum. In this respect the conditions in (1.16) differ essentially from the well-known stability requirement on motion in a magnetic field, $0 < n < 1$; the particle velocity does not enter in the magnetic case.

We may note that the relativistic effect in the dependence of $\omega_{r,z}/\omega$ on β can cause a significant change in the focusing properties of electrostatic lenses. For example, in electron optics it is generally assumed that in a cylindrical condenser ($n_e = 1$) the image is at an azimuthal angle of 127° with respect to the source (cf. for example reference 2). However, examination of Eq. (1.15) indicates that this relation is valid only in the nonrelativistic case, i.e. when $\beta = 0$. The angle indicated above changes at relatively small electron energies, becoming 180° when $\beta = 1$.

We now consider the synchrotron oscillations in an electric field; in some respects these also differ from the synchrotron oscillations in an axially symmetric magnetic field. In the expression for the frequency of synchroton oscillations,

$$\Omega^2 = q\omega^2 eV_0\sin\varphi_s K_1/2\pi E_s \qquad (1.17)$$

the quantity K_1 for the magnetic accelerator corresponds to the parameter K defined by the relations

$$\Delta\omega/\omega = -K\Delta E/E_s, \quad \Delta r/r_s = \alpha\Delta p/p_s, \quad K = 1-(1-\alpha)/\beta_s^2, \qquad (1.18)$$

where p is the particle momentum.

In the present case we have in place of the relation $K_1 = K$

$$K_1 = K/(1+\alpha), \quad \alpha = (2-\beta_s^2)/(1-n_e). \qquad (1.19)$$

We may note that in contrast with the magnetic case, where $\alpha = 1/(1-n_m)$, the quantity α depends on the particle velocity. It is apparent at "critical" energies (i.e., energies such that $E = E_{cr} = \sqrt{-n_e}E_0$) $K = 0$ and phase focusing no longer obtains [cf. Eq. (1.19)]. When $n_e > 0$ (i.e., when the electric field falls off radially) there is no critical energy.

2. MOTION OF PARTICLES IN AN AXIALLY SYMMETRIC FIELD WITH RADIATION EFFECTS TAKEN INTO ACCOUNT

In analyzing the motion of a relativistic electron in an electric field it is necessary to take account of radiation effects. We start with the procedure used in reference 3, analyzing the radiation effects within the framework of classical electrodynamics and assuming that in the equilibrium orbit the radiation losses are compensated by the electric accelerating field. In the right side of Eqs. (12) to (14) we add terms to take account of the radiation (the small parameter is $1/\gamma$ where $\gamma = 1/\sqrt{1-\beta_s^2}$. After some simple transformations we obtain the radial equation

$$\ddddot{x} + [2\dot{E}_s/E_s + 3W_s/E_s - 2\dot{\Omega}/\Omega]\dddot{x} + \omega^2(2-n_e)\ddot{x}$$
$$+ \omega^2[(2-n_e)(\dot{E}_s/E_s - 2\dot{\Omega}/\Omega) + (3-4n_e)W_s/E_s]\dot{x} \quad (2.1)$$
$$+ \omega^2\Omega^2(2-n_e)x = 0,$$

and the equation charcterizing the vertical oscillations

$$\ddot{z} + (\dot{E}_s/E_s + W_s/E_s)\dot{z} + \omega^2(n_e - 1)z = 0, \quad (2.2)$$

where W_s is the radiated power

$$W_s = (2e^4c/3E_0^4)E_s^2\mathscr{E}_0^2, \quad (2.3)$$

and Ω is given by Eq. (1.18).

In the first approximation we find that the radiation leads to damping of the oscillations (assuming compensation of the radiation losses); this damping is described by the expressions

$$a_{fr}(t) \sim E_s^{-1/2}\exp\left[-\frac{1}{2}\frac{3+n_e}{2-n_e}\int_0^t\frac{W_s}{E_s}d\xi\right], \quad (2.4)$$

$$a_{sr}(t) \sim E_s^{-1/2}\Omega^{1/2}\exp\left[-\frac{1}{2}\frac{3-4n_e}{2-n_e}\int_0^t\frac{W_s}{E_s}d\xi\right], \quad (2.5)$$

$$a_z(t) \sim E_s^{-1/2}\exp\left[-\frac{1}{2}\int_0^t\frac{W_s}{E_s}d\xi\right], \quad (2.6)$$

where a_{fr} and a_{sr} are respectively the amplitudes of the fast (betatron) and slow (synchrotron) oscillations.

These expressions are analogous to those obtained for motion in a magnetic field by Kolomenskii and Lebedev.[3] As in the magnetic case, the damping term, $a_z(t)$, is independent of n; this follows from the assumption of the existence of a plane of symmetry, $z = 0$. In order for the motion along r and z to be stable in the absence of radiation effects, the quantity n_e (for $\beta = 1$) must satisfy the condition

$$2 > n_e > 1. \quad (2.7)$$

On the other hand, it is apparent from Eqs. (2.4) and (2.5) that to have damping of the radial fast and slow oscillations the following condition must be satisfied when radiation is taken into account:

$$n_e < 3/4. \quad (2.8)$$

Obviously (2.7) and (2.8) cannot be satisfied simultaneously. Thus, the motion of an electron in an axially symmetric electric field is found to be unstable (if the radiation losses are compensated). Under certain conditions, however, stable motion can be achieved, for example, by means of an auxiliary axially symmetric magnetic field. If a magnetic field $H_z = H_{z0}(r_s/r)^{n_m}$ acts on the electron in addition to the electric field $\mathscr{E}_r = \mathscr{E}_{r0}(r_s/r)^{n_e}$, relativistic equilibrium is given by the relation:

$$eH_{z0}r_s = kE_s,\ e\mathscr{E}_{r0}r_s = (1-k)E_s,\ 0 \leqslant k \leqslant 1, \quad (2.9)$$

where k characterizes the relative effect of the magnetic field on the circular motion of the particle. When $k = 0$, the field is entirely electric; when $k = 1$, the field is entirely magnetic.

We introduce the effective index

$$n_{eff} = n_m k + n_e(1-k), \quad (2.10)$$

where by n_m and n_e we denote respectively the magnetic and electric field indices. It can be shown that if (2.9) is satisfied the equations of motion become

$$\ddddot{x} + \left(2\frac{E_s}{E_s} - 2\frac{\dot{\Omega}}{\Omega} + 3\frac{W_s}{E_e}\right)\dddot{x} + \omega^2(2-k-n_{eff})\ddot{x}$$

$$+ \omega^2\left[(2-k-n_{eff})\left(\frac{\dot{E}_s}{E_s} - 2\frac{\dot{\Omega}}{\Omega}\right) + (3-4n_{eff})\frac{W_s}{E_s}\right]\dot{x} \quad (2.11a)$$

$$+ \omega^2\Omega^2(2-k-n_{eff})x = 0$$

$$\ddot{z} + \left(\frac{\dot{E}_s}{E_s} + \frac{W_s}{E_s}\right)\dot{z} + \omega^2(n_{eff} - 1 + k)z = 0, \quad (2.12)$$

where

$$\Omega^2 = q\omega^2 eV_0\sin\varphi_s/2\pi E_s(2-k-n_{eff}).$$

The damping associated with compensated radiation is characterized by

$$\zeta_{fr}(t) \sim \exp\left[-\frac{1}{2}\frac{3-3k+n_{eff}}{2-k-n_{eff}}\int_0^t\frac{W_s}{E_s}d\xi\right], \quad (2.11b)$$

$$\zeta_{sr}(t) \sim \exp\left[-\frac{1}{2}\frac{3-4n_{eff}}{2-k-n_{eff}}\int_0^t\frac{W_s}{E_s}d\xi\right], \quad (2.13)$$

$$\zeta_z(t) \sim \exp\left[-\frac{1}{2}\int_0^t\frac{W_s}{E_s}d\xi\right]. \quad (2.14)$$

Since the frequency of the radial betatron oscillations is given by $\omega\sqrt{2-k-n_{eff}}$, the stability conditions (2.7) and (2.8) become

$$1-k < n_{eff} < 2-k, \quad (2.15)$$

$$n_{eff} < 3/4. \quad (2.16)$$

The inequalities in (2.15) and (2.16) yield

$$1-k<n_{\text{eff}}<3/4,\quad 1/4\leqslant k\leqslant 1. \tag{2.17}$$

In the n_m-n_e plane the region of stability lies between two parallel lines, one of which passes through the point (0, 1) and the other through the point ($\frac{3}{4}$, $\frac{3}{4}$). The width of the stability region is

$$d=\frac{3}{4}\cos\varphi\left(1-\frac{1}{3}\tan\varphi\right),$$

where

$$\varphi=\tan^{-1}\frac{1-k}{k}$$

is the slope of the lines with respect to the z axis. The maximum and minimum values of d are: $d_{max}=\frac{3}{4}$ (k = 1) and $d_{min}=0$ (k = $\frac{1}{4}$).

It is also possible to compute the effect of quantum radiation fluctuations on the motion of an electron in axially symmetric electric and magnetic fields. If the condition $E \ll E_{1/2}=m_0c^2[rm_0c/h]^{1/2}$ is satisfied this calculation can be carried out by the classical method used in reference 3. Carrying out these computations, we find the mean-square amplitude of the oscillations:

$$\langle a_{\text{fr}}^2\rangle=\frac{55}{24\sqrt{3}}\frac{b\Lambda\omega E_0}{(2-k-n_{\text{eff}})^2E_s}\times\int_0^t\exp\left[-\frac{n_{\text{eff}}+3-3k}{2-k-n_{\text{eff}}}\int_x^t\frac{W_s}{E_s}d\eta\right]\frac{E_s^3}{E_0^6}dx, \tag{2.18}$$

$$\langle a_{\text{sr}}^2\rangle=\frac{55}{24\sqrt{3}}\frac{b\Lambda\omega E_0^{1/2}V^{1/2}}{(2-k-n_{\text{eff}})^2E_s^{3/2}} \tag{2.19}$$

$$\times\int_0^t\exp\left[-\frac{3-4n_{\text{eff}}}{2-k-n_{\text{eff}}}\int_x^t\frac{W_s}{E_s}d\eta\right]\frac{E_s^{6.5}}{E_0^{5.5}}V^{-1/2}dx,$$

$$\langle a_z^2\rangle=\frac{13}{24\sqrt{3}}\frac{b\Lambda\omega E_s}{nE_0}\int_0^t\exp\left[-\int_x^t\frac{W_s}{E_s}d\eta\right]\frac{E_s^4}{E_0^4}dx, \tag{2.20}$$

where Λ is the Compton wave length and $b=e^2/m_0c^2$ is the classical radius of the electron. We may note that the mean-square vertical oscillations are given by an expression which formally is the same as that for motion in a magnetic field (without an electric field). This complete similarity does not exist for the radial oscillations, although the corresponding formulas are similar in many respects.

As in systems with magnetic guide fields, it is possible to use strong-focusing in electric guide fields; in this case n_e is a periodic (generally speaking, of alternating sign) function of azimuth $n_e(\vartheta+\vartheta_0)=n_e(\vartheta)$. Generally there is a difference from the motion in a magnetic field characterized by $n_m=n_m(\vartheta)$; this difference lies in the fact that the parameter $\mu_{x,z}$, which describes the frequency of the betatron oscillations (cf. reference 4), depends on the particle velocity. If, however, $|n_e|\gg 2$, these oscillations will be described in the same way as the betatron oscillations in magnetic strong-focusing systems. In particular, all results obtained for such systems when radiation effects are taken into account (cf. references 3, 5 and 6) can be carried over to the electric strong-focusing system.

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